113. *Analyticity of Complements of Complete Kähler Domains**)

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§1. **Statement of the result.** Let A be a real submanifold of a complex manifold M. We want to know the conditions on $X:=M-A$ which force A to be a complex submanifold of M. Our main result is the following

Theorem. *Under the above notations, assume that*

1) *X has a complete Kähler metric*

and that

2) *A is a regular submanifold of class C^1 with real codimension* 2. *Then A is a complex submanifold of M.*

Our theorem amounts to a partial answer to the following problem which was asked by T. Nishino.

Problem. Let $D \subset \boldsymbol{C}^n$ be a domain and $f: D \rightarrow \boldsymbol{C}$ a continuous function. Assume that there exists a plurisubharmonic function φ on a neighbourhood of $G(f):=\{(z', f(z'))\,;\, z' \in D\}$ such that $G(f)=\{z\,;\,\varphi(z)=-\infty\}$. *Is $G(f)$ a complex submanifold of $D \times \boldsymbol{C}$?*

I express sincere thanks to Dr. Y. Nishimura, who told me the problem and encouraged me.

§2. **Proof of the theorem.** Let X be a complex manifold of dimension n. X is called a complete Kähler manifold if X has a complete Kähler metric, i.e., a Kähler metric (of class C^2) which makes X a complete metric space.

Proposition (cf. Corollary (1.7) in [1]). *Let X be a complete Kähler manifold, φ a bounded strictly plurisubharmonic function of class C^4 on X and f a measurable $(n,1)$-form on X. Assume that f is square integrable with respect to the metric*

$$ds^2 := \sum_{\alpha,\beta} \frac{\partial^2 \varphi}{\partial z^\alpha \partial \bar{z}^\beta} dz^\alpha d\bar{z}^\beta,$$

where $(z^1, \cdots, z^n)$ denotes a local coordinate of X. Then there exists a square integrable $(n,0)$-form g on X satisfying $\bar{\partial} g = f$ if and only if $\bar{\partial} f = 0$.

Let M be a complex manifold containing X as a domain. We assume that $A:=M-X$ is a real two codimensional regular submanifold

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of class C^1. We are going to prove that A is a complex submanifold of M. Let $x \in A$ be any point. Then, by the above assumption, there exist a local coordinate $(z_1, \cdots, z_n)$ at x and a polydisc $U=\{|z_i|<1, 1 \leqq i \leqq n\}$ such that

i) $\pi_1(U-A) \cong \boldsymbol{Z}$

ii) $\{z_n=0\}$ meets $U \cap A$ transversally.

First we assume $n=2$. Then, choosing U smaller if necessary, we may assume moreover that $\Delta^* := \{z_2=0\} \cap (U-A)$ is biholomorphic to a punctured disc. Let $p: W \to U-A$ be the two-sheeted covering associated with the subgroup of index 2 of $\boldsymbol{Z}$. Note that $U-A$ and W are also complete Kähler manifolds. The holomorphic map $p: p^{-1}(\Delta^*) \to \Delta^*$ can be extended to a holomorphic map between unit discs. Let t be a holomorphic function on $p^{-1}(\Delta^*)$ satisfying $p^* z_1 = t^2$.

Let V_ε be a neighbourhood of Δ^* in U defined by

$$V_\varepsilon := \left\{ \left| \frac{z_2}{z_1} \right| < \varepsilon \right\}.$$

Since A and $\{z_2=0\}$ intersects only at x transversally, it follows that $V_\varepsilon \cap A = \emptyset$ for sufficiently small $\varepsilon > 0$. We fix such ε. Then we have a two-sheeted covering

$$p: p^{-1}(V_\varepsilon) \to V_\varepsilon.$$

The holomorphic retraction

$$\begin{array}{ccc} r: V_\varepsilon & \longrightarrow & \Delta^* \\ \cup\!\!\!\!\shortmid & & \cup\!\!\!\!\shortmid \\ (z_1, z_2) & \mapsto & (z_1, 0) \end{array}$$

can be lifted to a holomorphic retraction $\rho: p^{-1}(V_\varepsilon) \to p^{-1}(\Delta^*)$, so that $p \circ \rho = r \circ p$. We put $w_1 = \rho^* t$ and $w_2 = p^* z_2$. Then (w_1, w_2) is a coordinate on $p^{-1}(V_\varepsilon)$ and $w_1^2 = p^* z_1$.

We put

$$f = \begin{cases} \dfrac{\bar{\partial}(\chi(|w_2/w_1^2|^2) w_1^2)}{w_2} dw_1 \wedge dw_2, & \text{on } p^{-1}(V_\varepsilon), \\ 0, & \text{on } W - p^{-1}(V_\varepsilon). \end{cases}$$

Here, χ is a C^∞ function satisfying $\chi=1$ on $(-\infty, \varepsilon^2/2)$ and $\chi=0$ on (ε^2, ∞). We set $C = \sup |\chi'|$.

Assertion. *f is square integrable with respect to the metric $p^*(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2)$.*

Proof. On $p^{-1}(V_\varepsilon)$, we have

$$\begin{aligned} f &= \frac{w_1^2}{w_2} \bar{\partial} \chi\left(\left| \frac{w_2}{w_1^2} \right|^2 \right) \wedge dw_1 \wedge dw_2 \\ &= \chi'\left(\left| \frac{w_2}{w_1^2} \right|^2 \right) \left\{ \frac{1}{\bar{w}_1^2} d\bar{w}_2 \wedge dw_1 \wedge dw_2 - \frac{2\bar{w}_2}{\bar{w}_1^3} d\bar{w}_1 \wedge dw_1 \wedge dw_2 \right\}. \end{aligned}$$

Since $p^*(dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2) = |w_1|^2 dw_1 d\bar{w}_1 + dw_2 d\bar{w}_2$, we have

$$|f|^2 dv = \left| \chi'\left(\left| \frac{w_2}{w_1^2} \right|^2 \right) \right|^2 \left(\frac{1}{|w_1|^6} + \frac{4|w_2|^2}{|w_1|^8} \right) dv,$$

where $|f|$ and dv denote resp. the length of f and the volume form with respect to $p^*(dz_1d\bar{z}_1+dz_2d\bar{z}_2)$. Therefore

$$\int_W |f|^2 dv=\int_{p^{-1}(V_\varepsilon)} |f|^2 dv=\int_{p^{-1}(V_\varepsilon)} \left|\chi'\left(\left|\frac{w_2}{w_1^2}\right|^2\right)\right|^2\left(\frac{1}{|w_1|^6}+\frac{4\,|w_2|^2}{|w_1|^8}\right)dv$$
$$=2\int_{V_\varepsilon} \left|\chi'\left(\left|\frac{z_2}{z_1}\right|^2\right)\right|^2\left(\frac{1}{|z_1|^3}+\frac{4\,|z_2|^2}{|z_1|^4}\right)dv_*$$
$$=2\int_U \left|\chi'\left(\left|\frac{z_2}{z_1}\right|^2\right)\right|^2\left(\frac{1}{|z_1|^3}+\frac{4\,|z_2|^2}{|z_1|^4}\right)dv_*$$
$$=2\int_{|z_1|<1} \frac{1}{|z_1|^3}\left\{\int_{|z_2|<1} \left|\chi'\left(\left|\frac{z_2}{z_1}\right|^2\right)\right|^2\left(1+\frac{4\,|z_2|^2}{|z_1|}\right)dv_2\right\}dv_1.$$

Here $dv_*=-(1/4)dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2$ and $dv_i=(\sqrt{-1}/2)dz_i\wedge d\bar{z}_i$ $(i=1,2)$. On the support of $\chi(|z_2/z_1|^2)$, we have

$$|z_2|\leqq\varepsilon\,|z_1|.$$

Hence

$$\int_{|z_2|<1} \left|\chi'\left(\left|\frac{z_2}{z_1}\right|^2\right)\right|^2\left(1+\frac{4\,|z_2|^2}{|z_1|}\right)dv_2<C^2(1+4\varepsilon)\pi\varepsilon^2\,|z_1|^2.$$

Therefore

$$\int_{|z_1|<1} \frac{1}{|z_1|^3}\left\{\int_{|z_2|<1} \left|\chi'\left(\left|\frac{z_2}{z_1}\right|^2\right)\right|^2\left(1+\frac{4\,|z_2|^2}{|z_1|}\right)dv_2\right\}dv_1$$
$$<C^2(1+4\varepsilon)\pi\varepsilon^2\int_{|z_1|<1}\frac{1}{|z_1|}dv_1<\infty.$$

Q.E.D.

Thus, applying the proposition to $f, W(=X)$, and $p^*(|z_1|^2+|z_2|^2)$ $(=\varphi)$, we obtain a square integrable (2, 0)-form g on W satisfying $\bar{\partial}g=f$. We put

$$h=\begin{cases}\chi\left(\left|\dfrac{w_2}{w_1^2}\right|^2\right)w_1^2dw_1\wedge dw_2-w_2g, & \text{on } p^{-1}(V_\varepsilon),\\ -w_2g, & \text{on } W-p^{-1}(V_\varepsilon).\end{cases}$$

Then h is a square integrable holomorphic 2-form on W. Let σ be the covering transformation of $p: W\to U-A$. We set $h_*=(h-\sigma^*h)/2$. Then we have

$$h_*=\chi\left(\left|\frac{w_2}{w_1^2}\right|^2\right)w_1^2dw_1\wedge dw_2-w_2\left(\frac{g-\sigma^*g}{2}\right).$$

Since $p^*(dz_1d\bar{z}_1+dz_2d\bar{z}_2)$ is invariant under σ, h_* is square integrable, too. Therefore $F=h_*/p^*(dz_1\wedge dz_2)$ is a square integrable holomorphic function on W. Since $\sigma^*F=-F$, there exists a holomorphic function F_1 on $U-A$ satisfying $F^2=p^*F_1$. F_1z_1 is integrable with respect to $dz_1d\bar{z}_1+dz_2d\bar{z}_2$. Hence F_1 is a meromorphic function on U whose poles are contained in $A\cup\{z_1=0\}$. We set $F_1^0=\{z\in U; F_1(z)=0\}$ and $F_1^\infty=\{z\in U; F_1(z)=\infty\}$. Since $F_1|_{\Delta^*}=z_1|_{\Delta^*}$, there exists an irreducible component S of $F_1^0\cup F_1^\infty$ whose multiplicity is odd. Since $p^*F_1=F^2$, S must be contained in A. Thus A is a complex submanifold of M.

In the general case $n>2$, we use the following characterization of complex submanifolds.

Lemma. *Let A be a real C^1-submanifold of a complex manifold M. A is a complex submanifold of M if and only if the complex structure of TM (the tangent bundle of M) induces an almost complex structure in TA.*

Proof is trivial.

Let $v \in T_x A$ be any tangent vector. There exists a bidisc B intersecting A transversally and $v \in T_x(A \cap B)$. As we have shown, $A \cap B$ is a complex submanifold of B. Therefore $Jv \in T_x(A \cap B)$, where J denotes the complex structure of M. Thus, by the lemma, A is a complex submanifold of M. This completes the proof of the main theorem.

§ 3. **Remark.** In virtue of our theorem, Nishino's conjecture is true if f is C^1 and φ is C^4 outside $G(f)$. In fact, under these conditions, for any point $z' \in D$, there exists a Stein neighbourhood $U' \ni x$ such that $U_c := \{z \in U' \times \boldsymbol{C};\ -\infty < \varphi(z) < -c\}$ is provided with a complete Kähler metric defined by

$$\sum_{\alpha,\beta} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} \left\{ -\log(-\varphi) + \frac{1}{-c-\varphi} \right\} dz^\alpha d\bar{z}^\beta + q^* ds^2.$$

Here c is a sufficiently large number, q is the projection $U' \times \boldsymbol{C} \to U'$ and ds^2 is a complete Kähler metric on U'.

Reference

[1] T. Ohsawa: On complete Kähler domains with C^1-boundary. Publ. RIMS, Kyoto University, **16**, 929–940 (1980).