112. On Three-Dimensional Compact Complex Manifolds with Non-Positive Kodaira Dimension

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Introduction. The structure of algebraic threefolds with non-positive Kodaira dimension κ has been studied by Ueno [8], [9] and Viehweg [10]. Their results are based on the semi-positivity theorem of the direct image sheaf of the relative canonical sheaf of a fibre space ([4]). This is a consequence of the theory of variation of Hodge structure. Therefore it is easy to show that the similar results hold for compact Kähler manifolds of dimension three with $\kappa\leq 0$.

On the other hand, Atiyah [1] and Blanchard [2] showed that the semi-positivity theorem does not necessarily hold for non-Kähler fibre spaces. Hence it is expected that the structure of non-Kähler manifolds is different from that of Kähler manifolds.

The main purpose of the present note is to announce structure theorems of compact complex manifolds of dimension three with $\kappa\leq 0$ which have non-trivial Albanese tori. Contrary to the case of analytic surfaces, we have interesting new phenomena.

1. Preliminaries. In the present note, by an *analytic threefold* M we mean a compact complex manifold of dimension three. We use the following notation:

$$p_g(M)=h^0(M,K_M),\quad P_m(M)=h^0(M,K_M^m),\quad m=1,2,3,\cdots,$$
$$g_\nu(M)=h^0(M,\Omega_M^\nu),\qquad \nu=1,2,\cdots,\dim M,$$
$$q(M)=h^1(M,\mathcal{O}_M),$$

where K_M is the canonical bundle of M and Ω_M^ν is the sheaf of germs of holomorphic ν-forms on M. These are bimeromorphic invariants. Put $N(M)=\{m\geq 1\,|\,P_m(M)\geq 1\}$. The Kodaira dimension $\kappa(M)$ of M is defined by

$$\kappa(M)=\begin{cases}\max\limits_{m\in N(M)} \Phi_{mK}(M), & \text{if } N(M)\neq\emptyset\\ -\infty, & \text{if } N(M)=\emptyset,\end{cases}$$

where $\Phi_{mK}(M)$ is a meromorphic mapping of M into $\boldsymbol{P}^N(\boldsymbol{C})$, $N=P_m(M)-1$, associated with the m-canonical system $|mK_M|$ of M. Hence, $\kappa(M)=-\infty$ if and only if $P_m(M)=0$ for $m=1,2,\cdots$, and $\kappa(M)=0$ if and only if $P_m(M)\leq 1$ for $m=1,2,\cdots$, and the equality holds for some positive integer m.

According to Fujiki [3], we use the following

Definition 1. A compact complex manifold M is in a subcategory $\mathcal{C}$ of the category of compact complex manifolds if M is a meromorphic image of a compact Kähler manifold N.

Note that, in the above definition, $\dim N$ may be greater than $\dim M$. If a compact complex manifold M is in $\mathcal{C}$, then the Hodge decomposition theorem of the cohomology group $H^*(M, \boldsymbol{C})$ holds. (See [7], [9].)

With any compact complex manifold M we can associate the Albanese torus $A(M)$ and the Albanese mapping $\alpha: M \to A(M)$. ([2], [7], [9].) Put $t(M) = \dim A(M)$. The number $t(M)$ is called the *Albanese dimension* of M. It is a bimeromorphic invariant. By virtue of the construction of the Albanese torus ([2]), we have

$$t(M) \leq h^0(M, d\mathcal{O}_M) \leq g_1(M).$$

If M is in $\mathcal{C}$, the both equalities hold.

Note that for the Albanese mapping $\alpha: M \to A(M)$, fibres of $\alpha: M \to \alpha(M)$ may not be connected. Let

$$\begin{array}{ccc} \alpha: M & \longrightarrow & \alpha(M) \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\gamma} & \\ & W & \end{array}$$

be the Stein factorization of α so that $\beta: M \to W$ has connected fibres. The fibre space $\beta: M \to W$ is called the *Albanese fibration*.

Let us recall the bimeromorphic classification of analytic surfaces with $\kappa \leq 0$ due to Kodaira [5]. In the following table we assume that all surfaces contain no exceptional curves of the first kind.

Theorem. *If $\kappa(S) = 0$, $g_1(S) \geq 1$, then Albanese mapping $\alpha: S \to A(S)$ is surjective and has connected fibres. If $g_1(S) = 2$, then α is isomorphic. If $g_1(S) = 1$, then $\alpha: S \to A(S)$ has a structure of an analytic fibre bundle whose fibre is an elliptic curve.*

Table I. Classification table of analytic surfaces with $\kappa = 0$

b_1	q	g_1	p_g	P_{12}	structure
4	2	2	1	1	complex torus
3	2	1	1	1	elliptic surface with trivial canonical bundle
2	1	1	0	1	hyperelliptic surface
1	1	0	0	1	elliptic surface of class VII_0 with trivial K^m for a positive integer $m \geq 2$
0	0	1		1	$K3$ surface
		0		1	Enriques surface

The number $b_1(S)$ is the first Betti number of S. By definition, analytic surface S belongs to the class VII_0, if $b_1(S)=1$, $q(S)=1$, $p_g(S)=0$.

Table II. Classification table of analytic surfaces with $\kappa=-\infty$

b_1	q	g_1	structure
$2g$	$g\geqslant 1$	$g\geqslant 1$	P^1-bundle over a curve of genus g
1	1	0	surface of class VII_0
0	0	0	P^2 or P^1-bundle over P^1

Hence, if $\kappa(S)=-\infty$ and S does not belongs to the class VII_0, then S is algebraic and is rational or ruled.

2. Main Theorems. Let us state the structure theorems of analytic threefolds with $\kappa\leq 0$ which have non-trivial Albanese tori.

Main Theorem 1. *Let M be an analytic threefold with $\kappa(M)=0$ and $t(M)\geq 1$.*

1) *If Albanese mapping $\alpha: M\to A(M)$ is surjective, then α has connected fibres and M has the following properties.*

a) *If $t(M)=3$, then α is bimeromorphic.*

b) *If $t(M)=2$, then $\alpha: M\to A(M)$ is bimeromorphically equivalent to an analytic fibre bundle over $A(M)$ whose fibre is an elliptic curve.*

c) *If $t(M)=1$, then any smooth fibre M_x of the Albanese mapping α is a surface with $\kappa(M_x)=0$. Moreover, if M belongs to $\mathcal{C}$, then $\alpha: M\to A(M)$ is bimeromorphically equivalent to an analytic fibre bundle over $A(M)$ whose fibre is a surface with $\kappa=0$.*

2) *If the Albanese mapping α is not surjective, the image $\alpha(M)=C$ is a non-singular curve of genus $g\geq 2$. The Albanese mapping has connected fibres and a general fibre M_x of α is bimeromorphically equivalent to a complex torus, or a K3 surface. In this case, M does not belong to $\mathcal{C}$.*

Main Theorem 2. *Let M be an analytic threefold with $\kappa(M)=-\infty$ and $t(M)\geq 1$. Then, we have* $\dim\alpha(M)\leq 2$.

1) *If* $\dim\alpha(M)=2$, *then a general fibre of the Albanese fibration $\alpha: M\to S$ is P^1.*

2) *If* $\dim\alpha(M)=1$, *then the image $\alpha(M)=C$ of the Albanese mapping $\alpha: M\to A(M)$ is a non-singular curve, $\alpha: M\to C$ has connected fibres, and a general fibre M_x of α is a surfaces with $\kappa=-\infty$, or bimeromorphically equivalent to a complex torus or a K3 surface. Moreover, if M belongs to the class $\mathcal{C}$, general fibres of α are rational or ruled surfaces.*

Remark. 1) In Main Theorem 1, 1), c), if M does not belong to $\mathcal{C}$, then $\alpha: M\to A(M)$ is not necessarily bimeromorphically equivalent to an analytic fibre bundle.

2) In Main Theorem 1, 2), the curve C has arbitrary genus $g \geq 2$.

Example. Using Atiyah's method [1], we construct complex analytic families of analytic threefolds with $\kappa \leq 0$, which show strange phenomena of analytic threefolds.

Let $\pi: C \to \boldsymbol{P}^1$ be a double covering ramified at $2g-2$ points so that C is a hyperelliptic curve of genus g. Put $L = \pi^* \mathcal{O}_{P^1}(1)$, $F = L^l$. Hence $K_C = L^{g-1}$. For any point $t \in \mathrm{Pic}^0(C)$, we let $[t]$ be the line bundle of degree zero on C corresponding to the point t. Put $F_t = F \otimes [t]$. Let $\mathcal{F}$ be the line bundle over $C \times \mathrm{Pic}^0(C)$ such that the restriction $\mathcal{F}|_{C \times t}$ is isomorphic to F_t. Assume $m \geq 2g$. Then F_t is generated by its global sections and $h^0(F_t) = m - g + 1$, $h^1(F_t) = 0$. Hence $p_* \mathcal{F}$ is locally free where $p: C \times \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C)$ is the natural projection. Then there exists an open neighbourhood U of the origin of $\mathrm{Pic}^0(C)$ and two holomorphic sections φ, ψ of $\mathcal{F}$ over $p^{-1}(U)$ such that $\varphi_t = \varphi|_{p^{-1}(t)}$, $\psi_t = \psi|_{p^{-1}(t)}$, considered as elements of $H^0(C, F_t)$, have no common zero on C for any $t \in U$. Put

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad I_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since rank $(\sum_{i=1}^4 a_i I_i) \leq 1$ for $(a_1, a_2, a_3, a_4) \in \boldsymbol{R}^4$ implies $a_1 = a_2 = a_3 = a_4 = 0$, $\Lambda_t = \sum_{i=1}^4 \boldsymbol{Z} I_i \begin{pmatrix} \varphi_t \\ \psi_t \end{pmatrix}$ is a lattice of the fibre of $V_t = F_t \oplus F_t$ at each point t of C. Since Λ_t acts each fibre of V_t as translations, we have a quotient manifold $M_t = V_t / \Lambda_t$. Let $\pi_t: M_t \to C$ be the natural projection. From our construction it is easily shown that π_t is smooth and we have

$$\omega_{M_t/C} = K_{M_t} \otimes \pi_t^* K_C^{-1} = \pi_t^* F_t^{-2}.$$

Moreover $\mathfrak{M} = \bigcup_{t \in U} M_t$ is a complex analytic family over U. We let $f: \hat{C} \to C$ be a double covering ramified at the divisor (s) where s is a generic element of $H^0(C, L^{2k})$. Then $K_{\hat{C}} = f^* L^{g-1+k}$. Put $\hat{M}_t = M_t \times_C \hat{C}$. Then $\hat{\pi}_t: \hat{M}_t \to \hat{C}$ is a smooth morphism and we have

$$\omega_{\hat{M}_t/\hat{C}} = \hat{\pi}_t^* f^* F_t^{-2}, \qquad K_{\hat{M}_t} = \hat{\pi}_t^* f^* L^{g-1+k-2l}[-2t].$$

Hence, if we put $k = 2l - g + 1$, then we have

$$K_{M_t} = \pi_t^* f^* [-2t].$$

Put $U' = U \cap \mathrm{Pic}^0(M)_{\mathrm{tor}}$ where $\mathrm{Pic}^0(M)_{\mathrm{tor}}$ is the set of all points of finite order in $\mathrm{Pic}^0(M)$. The set U' is dense in U and so is $U^* = U - U'$. For each point $t \in U'$, we have $\kappa(M_t) = 0$, and for each point $u \in U$, we have $\kappa(M_u) = -\infty$. Since $\hat{\mathfrak{M}} = \bigcup_{t \in U} \hat{M}_t$ is a complex analytic family, this shows that the Kodaira dimension is not a deformation invariant. Moreover, for any point $t \in U'$,

$$P_m(M_t) = \begin{cases} 1, & \text{if } 2mt = 0 \text{ in } \mathrm{Pic}^0(C), \\ 0, & \text{otherwise.} \end{cases}$$

Hence plurigenera are not deformation invariant. (Cf. [6].) Moreover, for a sufficiently large positive integer n, we can find a sequence of points $x_n, x_{n+1}, x_{n+2}, \cdots$, in U which converge to the origen O such that

$$P_m(M_{x_k})=\begin{cases}0, & m<k\\ 1, & m=k.\end{cases}$$

Hence we have the following:

For compact complex threefolds with $\kappa=0$, *we cannot find a positive integer* m *such* $P_m=1$, *even if they are in the same complex analytic family.*

Note that, from Table I we infer $P_{12}(S)=1$ for any analytic surface with $\kappa(S)=0$.

3. Fibre space over a curve. To prove the above main theorems, the following theorem plays an important role.

Theorem. *Let* $\varphi: V\to C$ *be a surjective morphism from an analytic threefold* V *to a non-singular curve* C *with connected fibres. Then we have*

$$\kappa(V)\geq\kappa(V_x)+\kappa(C)$$

for a general fibre V_x *of* φ, *if* V_x *is not bimeromorphically equivalent to a complex torus nor a* $K3$ *surface.*

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