107. *Surgery of Domain and the Green's Function of the Laplacian*

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(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1980)

§ 1. **Introduction and results.** Let M be a bounded domain in $\mathbf{R}^m$ with smooth boundary. Let N be a compact connected regular smooth submanifold of M. We put $n=\dim N$. In this note we assume that $m\geq n+2\geq 3$. For any sufficiently small $\varepsilon>0$, let Γ_ε be the ε-tubular neighbourhood of N defined by

$$\Gamma_\varepsilon=\{x\in M;\ \mathrm{dist}(x, N)<\varepsilon\}.$$

We put $M_\varepsilon=M\setminus\bar{\Gamma}_\varepsilon$.

Let $G(x, y)$ (resp. $G_\varepsilon(x, y)$, $\varepsilon>0$) be the Green's function of the Laplacian with the Dirichlet condition on ∂M (resp. ∂M_ε).

In this paper we report the following theorems.

Theorem 1. *Assume* $m-n\geq 5$, *then for any fixed* $x, y\in M\setminus N$

$$G_\varepsilon(x, y)=G(x, y)-3S_5\varepsilon^{m-n-2}\int_N G(x, w)G(y, w)dw+O(\varepsilon^{m-n})$$

holds when ε *tends to zero. Here* S_k *denotes the area of the unit sphere in* $\boldsymbol{R}^k$.

Theorem 2. *Assume* $m-n=3$ or 4, *then for any fixed* $x, y\in M\setminus N$

$$G_\varepsilon(x, y)=G(x, y)-(m-n-2)S_{m-n}\varepsilon^{m-n-2}\int_N G(x, w)G(y, w)dw+O(K(\varepsilon))$$

holds when ε *tends to zero. Here* $K(\varepsilon)=\varepsilon^4|\log\varepsilon|$ *in case* $m-n=4$ *and* $K(\varepsilon)=\varepsilon^2$ *in case* $m-n=3$.

Theorem 3. *Assume* $m-n=2$, *then for any fixed* $x, y\in M\setminus N$

$$G_\varepsilon(x, y)=G(x, y)+(2\pi)(\log\varepsilon)^{-1}\int_N G(x, w)G(y, w)dw+O((\log\varepsilon)^{-2})$$

holds when ε *tends to zero.*

It should be remarked that the remainder terms $O(\varepsilon^{m-n})$, $O(K(\varepsilon))$, $O((\log\varepsilon)^{-2})$ in Theorems 1–3 are not uniform with respect to x, y.

Theorems 1–3 above are the versions of the Schiffer-Spencer formula which describes the asymptotic property of $G_\varepsilon(x, y)$ when ε tends to zero in case $m=2$ and $n=0$. See Schiffer-Spencer [2]. The author considers the case $m\geq 3$, $n=0$ in Ozawa [1].

Using the techniques developed in [1], we can get an asymptotic formula for eigenvalues of the Laplacian. We give some notations. Let $0>\lambda_1(\varepsilon)\geq\lambda_2(\varepsilon)\geq\cdots$ be the eigenvalues of the Laplacian in M_ε with the Dirichlet condition on ∂M_ε. And let $0>\lambda_1\geq\lambda_2\geq\cdots$ be the eigenvalues of the Laplacian in M with the Dirichlet condition on ∂M. We arrange them repeatedly according to their multiplicities.

We have the following

Theorem 4. *Assume $m=3$ and $n=1$. Fix j. Suppose that the multiplicity of λ_j is one, then*

$$\lambda_j(\varepsilon)=\lambda_j+2\pi(\log\varepsilon)^{-1}\int_N \varphi_j(w)^2dw+O((\log\varepsilon)^{-2})$$

holds when ε tends to zero. Here $\varphi_j(x)$ denotes the normalized eigenfunction of the Laplacian associated with λ_j.

When $j=1$, the above theorem gives a good asymptotic expression of the shift of the fundamental tone when a fine wire is placed in a region.

Details of the proofs of Theorems 1–4 will be given elsewhere.

§2. Sketch of proof of Theorem 1. Recall that $m-n\geq 5$. We fix $w\in N$. We know that $G(x,w)$ has the following asymptotic property when x tends to $w\in N$:

$$\lim_{x\to w}(G(x,w)-((m-2)S_m)^{-1}|x-w|^{-m+2})=C(w)$$

for some constant $C(w)$. Here S_m denotes the area of the unit sphere in R^m. Moreover, we can take $L>0$ such that

$$|G(x,w)-((m-2)S_m)^{-1}|x-w|^{-m+2}-C(w)|\leq L|x-w|$$

holds for any $x\in M\backslash N$, $w\in N$.

Let $w_1,\cdots,w_n,w_{n+1},\cdots,w_m$ be a fixed orthonormal coordinate system in R^m with the origin $\tilde{w}$. We assume that the subspace given by $\{(w_1,\cdots,w_n,0,\cdots,0);\ w_i\in R,\ 1\leq i\leq n\}$ is the tangent plane of N at $\tilde{w}$. We put

$$\langle f(\tilde{w}),g(\tilde{w})\rangle_{*w}=\sum_{k=n+1}^{m}\left(\frac{\partial f}{\partial w_k}\right)(\tilde{w})\left(\frac{\partial g}{\partial w_k}\right)(\tilde{w})$$

for any $f,\ g\in C^\infty(M)$.

Put

$$Q_\varepsilon(x,y)=G_\varepsilon(x,y)-\mathrm{G}(x,y)+C_{m,n}\varepsilon^{m-n-2}\int_N G(x,w)G(y,w)dw$$
$$+D_{m,n}\varepsilon^{m-n}\int_N\langle G(x,w),G(y,w)\rangle_{*w}dw$$

for any $x,\ y\in M\backslash N$. Here

$$C_{m,n}=2(m-2)S_mS_n^{-1}B(n/2,(m-n-2)/2)^{-1},$$
$$D_{m,n}=2S_mS_n^{-1}(B(n/2,(m-n)/2))^{-1}$$

where $B(p,q)$ denotes the beta function.

Fix $y\in M\backslash N$. Then it is easy to see that

$$\varDelta_xQ_\varepsilon(x,y)=0\qquad x\in M_\varepsilon$$

and

$$Q_\varepsilon(x,y)=0\qquad x\in\partial M$$

for any sufficiently small $\varepsilon>0$. We want to estimate the absolute value of

$$Q_\varepsilon(x, y)|_{x\in\partial\Gamma_\varepsilon}$$

from above.

We have the following

Lemma 1. *We assume that $m \geq n+5$. Let g be an arbitrary fixed smooth function on N. Fix an arbitrary $w^* \in N$. Then there exists a positive constant C independent of $w^* \in N$ such that*

$$(2.1)_m \qquad \left| \int_N \frac{g(w)}{|x-w|^{m-2}} dw - (S_n \cdot B(n/2, (m-n-2)/2)/2) \cdot g(w^*) \times |x-w^*|^{n-m+2} \right| \leq C|x-w^*|^{n-m+4}$$

holds when x tends to $w^* \in N$ along the normal directions to N at w^* with respect to M.

For any w and z contained in a fixed sufficiently small open neighbourhood of M, let $P(w \to z)$ denote the minimal geodesic curve starting from w through z.

Since we know the asymptotic properties of $G(x, w^*)$ and $\partial G(x, w^*)/\partial w_j$ $(j=n+1, \cdots, m)$ when x tends to w^* along the normal directions to N at w^* with respect to M, we see from $(2.1)_m$ and a variant of $(2.1)_{m+1}$ that

$$(2.2) \qquad Q_\varepsilon(x, y)|_{x=x_0\in\partial\Gamma_\varepsilon} = -G(x, y)|_{x=x_0\in\partial\Gamma_\varepsilon} + G(w^*, y) + \varepsilon\partial_{w\to x_0} G(w, y)|_{w=w^*} + O(\varepsilon^2)$$

holds for any sufficiently small $\varepsilon > 0$. Here $\partial_{w\to x_0}$ denotes the partial derivative along $P(w \to x_0)$ with respect to w.

Summing up these facts we get

$$Q_\varepsilon(x, y)|_{x\in\partial\Gamma_\varepsilon} = O(\varepsilon^2)$$

when ε tends to zero. Following Lemma 2 and the above estimate imply the desired result.

Lemma 2. *Assume that $m-n \geq 3$. For an arbitrary fixed $\varepsilon > 0$, let $u_\varepsilon(x)$ be the function harmonic in M_ε satisfying*

$$u_\varepsilon(x)|_{x\in\partial M} = 0$$

$$u_\varepsilon(x)|_{x\in\partial\Gamma_\varepsilon} = 1.$$

Then for any fixed $x \in M \setminus N$, we have $u_\varepsilon(x) = O(\varepsilon^{m-n-2})$ when $\varepsilon \to 0$.

References

[1] Ozawa, S.: Singular variation of domains and eigenvalues of the Laplacian (1980) (preprint).

[2] Schiffer, M., and Spencer, D. C.: Functionals of Finite Riemann Surfaces. Princeton Univ. Press, Princeton (1954).