104. *On Systems of Differential Equations of Order Two with Fixed Branch Points*

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§ 1. **Introduction.** It was shown by K. Okamoto ([7]) that the six equations of Painlevé are converted to differential systems

$$\text{(S)} \qquad \begin{cases} y' = P(x, y, z) \\ z' = Q(x, y, z), \end{cases}$$

where P and Q are polynomials in y and z with coefficients rational in x, and moreover ([8]) that one can take as (S) Hamiltonian systems: $P = \partial H/\partial z$, $Q = -\partial H/\partial y$. By Painlevé systems we mean the Hamiltonian systems of Okamoto form obtained from the Painlevé equations and the Painlevé systems will be denoted by $\mathrm{P_I}$–$\mathrm{P_{VI}}$ according to the usual order.

It is natural to propose the following problem: To determine differential systems of the form (S) with fixed branch points and find canonical forms for such systems except for those whose integration is reduced to that of first order equations and linear differential equations of third order. This problem was studied by several authors [1], [2], [3], [6], [9], [10], but their results are far from completion.

Let m be the maximum of degrees of the polynomials P and Q with respect to y and z. It is remarkable that for the Painlevé systems, we have

$$m = \begin{cases} 2 & \text{for } \mathrm{P_I}, \mathrm{P_{II}}, \mathrm{P_{IV}} \\ 3 & \text{for } \mathrm{P_{III}} \\ 4 & \text{for } \mathrm{P_V}, \mathrm{P_{VI}} \end{cases}$$

and the degree of P with respect to z is one.

The purpose of this note is to announce the following

Theorem. *Suppose that*

1) *P and Q are polynomials in y, z with coefficients analytic in x,*

2) $2 \leqq m \leqq 5$,

3) *System* (S) *has no movable branch points,*

4) *the integration of* (S) *is not reducible to that of first order equations nor reducible to that of linear differential equations of third order.*

Then System (S) *is transformed to one of the Painlevé systems*

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$$\begin{cases} \dfrac{d\xi}{dt} = P_*(t, \xi, \eta) \\ \dfrac{d\eta}{dt} = Q_*(t, \xi, \eta) \end{cases}$$

by a change of variables

$$t = \chi(x), \quad \xi = \varphi(x, y, z), \quad \eta = \psi(x, y, z),$$

where for a generic value of x, the transformation

$$\xi = \varphi(x, y, z) \quad \textit{and} \quad \eta = \psi(x, y, z)$$

is a birational one.

The Painlevé systems $\mathrm{P_I}$, $\mathrm{P_{II}}$ *and* $\mathrm{P_{IV}}$ *are derived in the case* $m=2$, *the Painlevé systems* $\mathrm{P_I}$–$\mathrm{P_V}$ *in the case* $m=3$, *and all the Painlevé systems in the case* $m=4, 5$.

The theorem is proved only by the use of necessary conditions for System (S) to have fixed branch points, but the proof requires the examination of large number of cases.

We remark that System (S) of degree 5 is reduced to polynomial systems of degree $\leqq 4$.

§ 2. **Explanation of our methods.** Since P and Q are polynomials in y and z, System (S) can be regarded as a complex non-autonomous system whose phase space is the product of two copies of the complex projective line $\boldsymbol{P}^1$ for a generic value of x. It is clear that all singular points of System (S) are situated on the lines $y=\infty$ and $z=\infty$ for a generic x. We put

$$P(x, y, z) = P_m(x, y, z) + \cdots + P_1(x, y, z) + P_0(x)$$
$$Q(x, y, z) = Q_m(x, y, z) + \cdots + Q_1(x, y, z) + Q_0(x)$$

where P_k and Q_k are homogeneous polynomials in y and z of degree k, and $P_m \neq 0$ or $Q_m \neq 0$.

To seek necessary conditions for (S) to have fixed branch points, the following four methods are used repeatedly.

1) Blow-up of phase spaces.
2) Painlevé method of introducing a parameter (see [4]).
3) Application of a theorem of Briot-Bouquet (see [4]).
4) Application of a result of Malmquist ([5]).

We want to explain these methods by showing first stages of the proof.

First we blow up the space $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at the singularity $(y, z) = (\infty, \infty)$ for a generic x. For convenience we carry out the blow-up by putting

$$y = 1/y_1,\ z = z_1/y_1 \quad \text{and} \quad y = y_1/z_1,\ z = 1/z_1.$$

System (S) is taken by the first change of variables into a system of the form

$$\text{(1.1)} \qquad \begin{cases} y_1^{m-2} y_1' = -P_m(x, 1, z_1) + \cdots \\ y_1^{m-1} z_1' = Q_m(x, 1, z_1) - z_1 P_m(x, 1, z_1) + \cdots, \end{cases}$$

where $\cdots$ denotes terms containing y_1 as factor. Secondly, we introduce a parameter ε by making the change of variables

$$x=x_0+\varepsilon^{m-1}X, \quad y_1=\varepsilon Y, \quad z_1=Z,$$

where x_0 is a fixed generic point. Then the system (1.1) becomes

$$(1.2) \qquad \begin{cases} Y^{m-2}dY/dX=-P_m(x_0,1,Z)+\cdots \\ Y^{m-1}dZ/dX=Q_m(x_0,1,Z)-ZP_m(x_0,1,Z)+\cdots, \end{cases}$$

where $\cdots$ denotes terms containing ε as factor. A fundamental theorem due to Painlevé says that if the system (1.2) is a system with fixed branch points for $\varepsilon\neq 0$, so is the reduced system

$$(1.3) \qquad \begin{cases} Y^{m-2}dY/dX=-P_m(x_0,1,Z) \\ Y^{m-1}dZ/dX=Q_m(x_0,1,Z)-ZP_m(x_0,1,Z). \end{cases}$$

Applying the Painlevé theorem to (1.3) again, we see that the rational function $P_m(x_0,1,Z)/(Q_m(x_0,1,Z)-ZP_m(x_0,1,Z))$ has only simple poles. From this fact and the system (1.3), we obtain an equation of the form

$$(1.4) \qquad dZ/dX=A\Pi(Z-a_j)^{\mu_j},$$

where $A, a_1, a_2, \cdots$ and $\mu_1, \mu_2, \cdots$ are constants. The Briot-Bouquet theorem gives a complete list of types of the equation (1.4) under the condition that all solutions of (1.4) are single-valued. The combination of the use of Briot-Bouquet theorem and the relation

$$Y^{m-1}=(Q_m(x_0,1,Z)-ZP_m(x_0,1,Z))/dZ/dX$$

leads us to complete determination of P_m and Q_m. The method explained above is very powerful to obtain necessary conditions on some coefficients in P and Q.

To search for other conditions on coefficients in the case when $m=2,3,4$, we need the following Malmquist result. After a finite number of blow-ups, we encounter systems of the form

$$(1.5) \qquad \begin{cases} dy/dx=a(x)+[x;y,z]_1 \\ ydz/dx=b(x)z+c(x)y+[x;y,z]_2 \end{cases}$$

where a and b are functions holomorphic and non-vanishing in a domain and $b(x)/a(x)$ is equal to a positive integer n identically, and $[x;y,z]_k$ denotes a convergent power series in y, z with coefficients holomorphic in x of total order k. We rewrite (1.5) as

$$(1.6) \qquad \begin{cases} dx/dy=A(x)+[x,y,z]_1 \\ ydz/dy=nz+C(x)y+[x,y,z]_2. \end{cases}$$

Malmquist showed that there exists a transformation

$$\begin{cases} x=u+\sum p_{jk}(u)y^j\zeta^k \\ z=\zeta+\sum q_{jk}(u)y^j\zeta^k \end{cases}$$

which takes (1.6) into a system

$$\begin{cases} du/dy=0 \\ \zeta d\zeta/dy=n\zeta+D(u)y^n, \end{cases}$$

where the p_{jk} and q_{jk} are holomorphic and the series $\sum p_{jk}(u)y^j\zeta^k$ and $\sum q_{jk}(u)y^j\zeta^k$ are convergent ones with $p_{10}(u)=A(u)$. It follows from this result that $D(u)\equiv 0$ is a necessary condition for (S) to have fixed branch points.

§3. Brief outline of the proof in the case $m=4$. We shall give a brief sketch of the proof in the case $m=4$. From the determination of P_4 and Q_4 we have only to examine the following cases:

1) $P_4=0,\quad Q_4=b(x)y^4,$
2) $P_4=0,\quad Q_4=b(x)y^3z,$
3) $P_4=(1+(1/\nu))a(x)y^3z,\quad Q_4=-(2-(1/\nu))a(x)y^2z^2,$
$(\nu=\pm 1, \pm 2, \cdots, \infty)$
4) $P_4=0,\quad Q_4=b(x)y^2z(z-y),$
5) $P_4=a(x)y^2z(z-y),\quad Q_4=-a(x)yz^2(z-y).$

The use of the first three methods stated in §2 leads us to the following assertion: The cases 1) and 2) are excluded by the assumption 4) of the theorem, and the cases 4), 5) by the assumption 3) of the theorem.

Now we turn to the case 3). The combination of the first three methods shows that we have a system of the form

$$\begin{cases} y'=p_1(x,y)z+p_2(x,y) \\ z'=q_1(x,y)z^2+q_2(x,y)z+q_3(x,y), \end{cases}$$

where p_1, p_2 are polynomials in y of degree 3 and q_1, q_2, q_3 are polynomials in y of degree 2. We distinguish the following three cases

3-1) $p_1=0$ has three distinct roots for a generic x,
3-2) $p_1=0$ has two distinct roots for a generic x,
3-3) $p_1=0$ has only one root for a generic x.

The application of the four methods implies that either we have $\nu=5$ and the polynomials p_2, q_2, q_3 are of degrees 2, 1, 0 respectively, or System (S) is reduced to a system of degree <4. Consider the first interesting case. More detailed considerations give us relations between coefficients of p_2, q_2, q_3. Using these relations and making a change of variables of the type stated in the theorem, we obtain the following conclusion:

System (S) is changed into the Painlevé system $\mathrm{P_{VI}}, \mathrm{P_V}$ or $\mathrm{P_{IV}}$ according to the case 3-1), 3-2) or 3-3), if System (S) is not reduced to a system of degree $\leqq 3$.

We want to give a remark for the case $m=3$. In this case we obtain a system

$$\begin{cases} y'=p_1(x,y)z+p_2(x,y) \\ z'=q_1(x,y)z^2+q_2(x,y)z+q_3(x,y) \end{cases}$$

where p_1, p_2 are polynomials in y of degree 2 and q_1, q_2, q_3 are polynomials in y of degree 1. The Painlevé system (V) or (III) is derived according as the equation $p_1(x,y)=0$ has two distinct roots or only one root for a generic x.

§4. An example and a conjecture. For a special case of System (S) of degree 6, we have

$$y'=\partial H/\partial z, \qquad z'=-\partial H/\partial y$$

with

$$x(x-1)H=y^4z^3+(\alpha y^3-(x+1)y^2)z^2+(\beta y^2+(\delta x-\delta-\alpha)y+x)z+\gamma y$$

where α, β, γ and δ are constants. It was shown by K. Okamoto that this system is transformed into the Painlevé system P_{VI} by the change of variables

$$y=1/v, \qquad z=av+v^2u$$

where a is any one of roots of the equation $X^3+\alpha X^2+\beta X+\gamma=0$.

To conclude, we want to propose the following conjecture: A polynomial system of the form (S) is equivalent to one of the Painlevé systems, if the system satisfies the conditions 1), 3) and 4) of the theorem.

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