# 103. *On Certain Numerical Invariants of Mappings over Finite Fields. III*

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**Introduction.** This is again a continuation of my two preceding papers*) [3]. We shall be concerned with algebras with involution and Hopf maps.

**§ 1. Algebras with involution.** Let $K=F_q$ ($q$ : odd) and let $A$ be an associative algebra with involution $\alpha$. (See [1] for basic facts on such algebras). Take an invertible element $\theta \in A$ such that

(1.1) $\theta^\alpha=\varepsilon\theta, \quad \varepsilon=\pm 1$

and consider the mapping $F: A \to A$ given by

(1.2) $F(x)=x^\alpha\theta x, \quad x \in A.$

Clearly, $F$ is a quadratic mapping of the underlying vector space of $A$ into itself. In this section, we shall determine invariants $\rho_F$, $\sigma_F$ for this mapping when the algebra $(A, \alpha)$ is simple. Since all finite division rings are commutative, there are 4 types of such algebras, up to the change of ground fields:

( i ) $A=K_r \oplus K_r, \quad (x, y)^\alpha=({}^t y, {}^t x), \quad \tau(x, y)=\mathrm{tr}\,(x)+\mathrm{tr}\,(y),$

(ii) $A=K_r, \quad x^\alpha=S^{-1}\,{}^t xS, \quad {}^t S=S, \quad \tau(x)=\mathrm{tr}\,(x),$

(iii) $A=K_{2s}, \quad x^\alpha=J^{-1}\,{}^t xJ, \quad J=\begin{pmatrix} 0 & 1_s \\ -1_s & 0 \end{pmatrix}, \quad \tau(x)=\mathrm{tr}\,(x),$

(iv) $A=L_r, \quad L=F_{q^2}, \quad x^\alpha=S^{-1}\,{}^t\bar{x}S, \quad {}^t\bar{S}=S, \quad \tau(x)=\mathrm{tr}\,(x)+\overline{\mathrm{tr}\,(x)}.$

(Here $\tau$ means the reduced trace of the algebra $A$ over $K$, $\mathrm{tr}\,(x)$ means the trace of the matrix $x$ and the bar means the conjugation of the quadratic extension $L/K$.) Note that the trace has the properties:

(1.3) $\tau(x^\alpha)=\tau(x)$, $\tau(xy)=\tau(yx)$, the mapping $(x, y) \mapsto \tau(x, y)$ is a non-degenerate symmetric bilinear form on $A$.

Therefore, to each $\lambda \in A^*$, the dual space of $A$, there corresponds uniquely an element $a=a_\lambda \in A$ such that $\lambda(x)=\tau(ax)$. Conversely, any $a \in A$ defines a linear form $\lambda=\lambda_a$ by $\lambda(x)=\tau(ax)$. We have

(1.4) $F_\lambda(x)=\lambda(F(x))=\tau(ax^\alpha\theta x).$

Put

(1.5) $\langle x, y\rangle_\lambda=\dfrac{1}{2}(F_\lambda(x+y)-F_\lambda(x)-F_\lambda(y)).$

Then, we have

*) As in my former paper (II), (I. 2.3) will mean (2.3) in (I).

(1.6) $r_\lambda = \operatorname{rank} F_\lambda = \dim A - \dim I_\lambda, \quad I_\lambda = \{x \in A\,;\ \langle x, y\rangle_\lambda = 0$ for all $y \in A\}$.

A simple computation using (1.3) shows that

(1.7) $\langle x, y\rangle_\lambda = \dfrac{1}{2}\tau((ax^\alpha\theta + \varepsilon a^\alpha x^\alpha\theta)y)$.

Hence, by (1.3), (1.6), we have

(1.8) $x \in I_\lambda \Leftrightarrow ax^\alpha + \varepsilon a^\alpha x^\alpha = 0 \Leftrightarrow x(a^\alpha + \varepsilon a) = 0$,

which, in particular, shows that $I_\lambda$ is a left ideal of $A$. Now, remember that only $\lambda$'s for which $r_\lambda$ is odd are meaningful for the computation of $\rho_F$ (see (II. 1.4)). Since every left ideal of our algebra $A$ is a direct sum of minimal left ideals whose dimensions are easily determined, we see already from (1.6) that $\rho_F = 0$ in the following cases: (i) $r$: even, (ii) $r$: even, (iii) and (iv). Therefore, it remains to consider the cases: (i) $r$: odd, (ii) $r$: odd.

Case (i) $r$: odd. If $\lambda = \lambda_c$ with $c = (a, b) \in A$, we have

(1.9) $I_\lambda = \{z = (x, y) \in A\,;\ z(c^\alpha + \varepsilon c) = 0\}$.

If we put $h = {}^t b + \varepsilon a$, then

(1.10) $I_\lambda = \{(x, y) \in K_r \oplus K_r\,;\ xh = y\,{}^t h = 0\} = M \oplus N$,

where $M = \{x \in K_r\,;\ xh = 0\}$, $N = \{y \in K_r\,;\ y\,{}^t h = 0\}$. If $\operatorname{rank} h = d$, then, normalizing $h$ by multiplying non-singular matrices on both sides, we see that $\dim M = r(r-d)$. Since rank ${}^t h = d$, it follows that $\dim I_\lambda = 2r(r-d)$ is even as well as $\dim A = 2r^2$, and we have $\rho_F = 0$, again.

Case (ii) $r$: odd. In this case, $A = K_r$, $r$: odd, $a^\alpha = S^{-1}\,{}^t a S$, ${}^t S = S$ and

(1.11) $I_\lambda = \{x \in A\,;\ x(a^\alpha + \varepsilon a) = 0\}, \qquad \varepsilon = \pm 1$.

As above, we see that $\dim I_\lambda = r(r-d)$ if $d = \operatorname{rank}(a^\alpha + \varepsilon a) = \operatorname{rank}({}^t(Sa) + \varepsilon(Sa))$, and so $r_\lambda = \dim A - \dim I_\lambda = rd$. Hence, only the case where $d$ is odd is meaningful. If $\varepsilon = -1$, $d$ is even because ${}^t(Sa) - (Sa)$ is skew-symmetric and we have $\rho_F = 0$, again. Therefore, we only have to consider the case $\varepsilon = 1$. We have then, by (II. 1.4),

(1.12) $\rho_F = (q-1)\sum_{r_\lambda \text{ odd}} q^{r^2 - r_\lambda} = (q-1)\sum_{\substack{1 \le d \le r \\ d \text{ odd}}} N_d q^{rd}$,

where $N_d$ means the cardinality of the set

(1.13) $E(r, d) = \{a \in K_r\,;\ \operatorname{rank}({}^t a + a) = d\}$, $d$: odd.

Along with the set (1.13), we need the set

(1.14) $S(r, d) = \{x \in A\,;\ {}^t x = x,\ \operatorname{rank} x = d\}$.

Clearly, the mapping $f: E(r, d) \to S(r, d)$ defined by $f(a) = {}^t a + a$ is a surjective mapping where each fibre consists of the same number $(= q^{(r(r-1))/2})$ of matrices, i.e. of all skew-symmetric matrices of degree $r$. (In fact, $f(a) = f(b) \Leftrightarrow b = a + c$, ${}^t c + c = 0$.) Therefore, we have

(1.15) $[E(r, d)] = q^{(r(r-1))/2}[S(r, d)]$.

As is well-known, every symmetric matrix of rank $d$ is congruent to either $P=\begin{pmatrix} 1_d & 0 \\ 0 & 0 \end{pmatrix}$ or $Q=\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$, where $R=\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \gamma \end{pmatrix}$, $\gamma$ being an element of $K^\times$ but not in $(K^\times)^2$. Call $G_P, G_Q$ the isotropy group of $P, Q$, respectively. Then, we have

(1.16) $[S(r,d)]=[GL_r(K)]/[G_P]+[GL_r(K)]/[G_Q]$.

Since we have

$$G_P=\left\{\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}\in K_r\,;\ X\in O(1_d),\ Y\in K_{r-d,d},\ Z\in GL_{r-d}(K)\right\}\quad\text{and}$$

$$G_Q=\left\{\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}\in K_r\,;\ X\in O(R),\ Y\in K_{r-d,d},\ Z\in GL_{r-d}(K)\right\},$$

(1.16) becomes

$$(1.17)\quad [S(r,d)]=\frac{[GL_r(K)]}{[O(1_d)][GL_{r-d}(K)]q^{(r-d)d}}+\frac{[GL_r(K)]}{[O(R)][GL_{r-d}(K)]q^{(r-d)d}}.$$

Consider, now, the polynomial $F_N(X)=(X-1)(X^2-1)\cdots(X^N-1)$. It is well-known that

(1.18) $[GL_N(K)]=q^{(N(N-1))/2}F_N(q)$.

(As for the cardinalities of geometric objects over $F_q$, see [2].) Let $g(r,d)$ be the cardinality of the set of $K$-rational points of grassmann variety of the vector space of dimension $r$ consisting of subspaces of dimension $d$. Then, we know that

$$(1.19)\quad g(r,d)=\frac{F_r(q)}{F_d(q)F_{r-d}(q)}.$$

Since $d$ is odd, we have

(1.20) $[O(1_d)]=[O(R)]=2q(q^2-1)q^3(q^4-1)\cdots q^{d-2}(q^{d-1}-1)$,

and it follows from (1.17), (1.19), (1.20) that

$$(1.21)\quad [S(r,d)]=g(r,d)\frac{[GL_d(K)]}{[O^+(1_d)]}=g(r,d)q^{(d^2-1)/4}(q-1)(q^3-1)\cdots(q^d-1).$$

Combining (1.12), (1.15), (1.21), we get

$$(1.22)\quad \rho_F=(q-1)q^{(r(r-1))/2}\sum_{\substack{1\leq d\leq r\\ d\text{ odd}}} g(r,d)q^{(d^2-1)/4}(q-1)(q^3-1)\cdots(q^d-1).$$

To sum up,

(1.23) **Theorem.** *Let $K=F_q$, $q$ : odd, $(A,\alpha)$ be one of algebras with involution over $K$ given by* (i), (ii), (iii), (iv) *and $F$ be the quadratic mapping $A\to A$ given by* (1.2). *Then, we have $\rho_F=0$ except for the case* (ii) $r$ : odd, $\varepsilon=1$, *and in this case $\rho_F$ is given by the formula* (1.22).

§ 2. **Hopf maps.** I would like to remark that we can obtain $\rho_F$ for a certain Hopf map $F$ as an application of the preceding section.

Consider an algebra $(A,\alpha)$ of type (ii) with $A=K_2$,

$$x^\alpha \equiv \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} \quad \text{when} \quad x=\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \quad \text{and} \quad \theta=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since $\theta^\alpha=-\theta$, we have $\varepsilon=-1$. The quadratic map

$$F(x)=x^\alpha\theta x=\begin{pmatrix} x_1x_2+x_3x_4 & x_2^2+x_4^2 \\ -(x_1^2+x_3^2) & -(x_1x_2+x_3x_4) \end{pmatrix}$$

sends $A=K_2=K^4$ into the subspace $K^3\subset A$ of matrices of trace 0. Furthermore, if we put $Q(x)=\det x=x_1x_4-x_2x_3$, then we have the relation $Q(F(x))=Q(x)^2$ which shows that the map $F: K^4\to K^3$ is a Hopf map. Since $\rho_F$ is independent of the embedding of the image of $F$ (see (I. 2.2)), (1.23) implies that $\rho_F=0$ for this Hopf map. Although we cannot develop here full story of Hopf maps (and non-associative algebras with involution as well), we hope to come back to it sometime, somewhere.

## References

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[3] Ono, T.: On certain numerical invariants of mappings over finite fields. I, II. Proc. Japan Acad., **56**A, 342–347; ibid., **56**A, 397–400 (1980).