102. *On the Regularity of Arithmetic Multiplicative Functions. I*

By J.-L. MAUCLAIRE*) and Leo MURATA**)

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 12, 1980)

1. Statement of result. An arithmetical function $f(n)$ is called *additive* (resp. *multiplicative*), if $f(mn)=f(m)+f(n)$ (resp. $f(mn)=f(m)f(n)$) for any pair m, n of relatively prime natural numbers, and is called *completely additive* (resp. *completely multiplicative*), if the above equality holds for every pair m, n.

We have several sufficient conditions under which an additive arithmetical function turns out to be completely additive. P. Erdös ([1]) proved that an additive arithmetical function which satisfies the condition

$$\lim_{n\to\infty}\{f(n+1)-f(n)\}=0, \tag{1.1}$$

is completely additive (and, more than that, it is equal to $c(\log n)$ for some c).

I. Kátai ([2]) succeeded in replacing (1.1) by a weaker condition,

$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leqslant x}|f(n+1)-f(n)|=0. \tag{1.2}$$

F. Skof ([3]) gave another condition: an additive arithmetical function which satisfies

$$\lim_{\substack{n\to\infty\\ n\notin S}}\{f(n+1)-f(n)\}=0, \tag{1.3}$$

where S is a sequence of density zero, is also completely additive.

We shall prove here a theorem which will show clearly the link between F. Skof's and I. Kátai's results.

Theorem. *Suppose $g(n)$ is a multiplicative arithmetical function such that:*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\leqslant x\\ n\notin S}}|g(n+1)-g(n)|=0, \quad \text{and} \quad |g(n)|=1, \tag{1.4}$$

where S is a sequence of density zero. Then $g(n)$ is completely multiplicative.

2. Proof of the theorem. Lemma. *Let $\{t_n\}_{n=1}^{\infty}$ be an increasing sequence of integers. Suppose $\limsup_{n\to\infty} t_n/n<\infty$, then, under the assumption of our theorem, we get*

---

*) C.N.R.S. (France) and Institute of Statistical Mathematics, Tokyo.
**) Department of Mathematics, Tokyo Metropolitan University.

$$\sum_{n\leqslant x}|g(t_n+1)-g(t_n)|=o(x).$$

Proof of the lemma. It is sufficient to prove

$$\sum_{n\leqslant x}|g(n+1)-g(n)|=o(x).$$

Since $|g(n+1)-g(n)|\leqslant 2$ for any n,

$$\sum_{\substack{n\leqslant x\\ n\in S}}|g(n+1)-g(n)|\leqslant 2\sum_{\substack{n\leqslant x\\ n\in S}}1=o(x).$$

This and (1.4) complete the proof.

Put $S_k(q)=(q^k-1)/(q-1)=q^{k-1}+\cdots+1$, where q is an even integer and k is a positive integer, and consider the sequence $\{q^k m+S_k(q)\}_{m=1}^{\infty}$. Since q and $q^{k-1}m+S_{k-1}(q)$ are relatively prime if $k\geqslant 2$, we have

$$g(q^k m+S_k(q)-1)=g(q)g(q^{k-1}m+S_{k-1}(q)),$$

and so from our lemma follows:

$$\sum_{m\leqslant x}|g(q^k m+S_k(q))-g(q)g(q^{k-1}m+S_{k-1}(q))|=o(x),$$

$(k\geqslant 2)$. From this follows

(2.1) $$\sum_{m\leqslant x}|g(q^s m+S_s(q))-g(q)^{s-1}g(qm)|=o(x),$$

where s is an arbitrary positive integer, because

$$\begin{aligned}&\sum_{m\leqslant x}|g(q^s m+S_s(q))-g(q)^{s-1}g(qm)|\\ &\leqslant\sum_{m\leqslant x}\sum_{k=2}^{s}|\{g(q^k m+S_k(q))-g(q)g(q^{k-1}m+S_{k-1}(q))\}g(q)^{s-k}|\\ &=o(x).\end{aligned}$$

Now we set $m=S_s(q)\{S_s(q)qn+1\}$, $n\geqslant 1$. It is easily seen that $S_s(q)$ and $q^s(S_s(q)qn+1)+1$ are relatively prime and so are any two of the three numbers, q^s, $S_s(q)$ and $S_s(q)qn+1$. Then, we obtain

$$\begin{aligned}g(q^s m+S_s(q))&=g\{q^s(S_s(q))(S_s(q)qn+1)+S_s(q)\}\\ &=g(S_s(q))g\{q^s(S_s(q)qn+1)+1\},\end{aligned}$$

and

$$\begin{aligned}g(q)^{s-1}g(qm)&=g(q)^{s-1}g\{q(S_s(q))(S_s(q)qn+1)\}\\ &=g(q)^s g(S_s(q))g(S_s(q)qn+1).\end{aligned}$$

From (2.1) follows then

(2.2) $$\sum_{n\leqslant x}|g\{q^s(S_s(q)qn+1)+1\}-g(S_s(q)qn+1)g(q)^s|=o(x).$$

But, since our lemma shows also that

$$\sum_{n\leqslant x}|g\{q^s(S_s(q)qn+1)+1\}-g(q^s(S_s(q)qn+1))|=o(x),$$

we get from (2.2) that

$$\sum_{n\leqslant x}|g(q^s)-g(q)^s|\,|g\{q^s(S_s(q)qn+1)\}|=o(x).$$

Consequently, since $|g(n)|=1$ for any integer, we get

$$\sum_{n\leqslant x}|g(q^s)-g(q)^s|=o(x),$$

and this means: $g(q^s)=g(q)^s$ for any even integer q and for any positive integer s. Put $q=2$, then $g(2^s)=g(2)^s$. And put $q=2p$, where p is an odd prime, then we have $g((2p)^s)=g(2p)^s$, and since $|g(2)|\neq 0$, $g(p^s)=g(p)^s$. This completes the proof of the theorem.

From this theorem, we can deduce the following corollary, which includes the result of F. Skof's as a special case and is closely connected to the problem solved by I. Kátai.

Corollary. *Suppose $f(n)$ is a complex-valued additive arithmetical function, S is a sequence of density zero, and*

$$\lim_{x\to\infty}\frac{1}{x}\sum_{\substack{n\leqslant x\\ n\notin S}}|f(n+1)-f(n)|=0. \tag{2.3}$$

Then $f(n)$ is completely additive.

Proof. Since $\mathbf{Re}\,(f(n))$, $\mathbf{Im}\,(f(n))$ are also additive and satisfy the similar condition as (2.3), it is sufficient to deal with a real-valued additive function $f(n)$. Put, for z real,

$$\exp\,(izf(n))=g(n,z).$$

Then by our theorem and the inequality,

$$|e^{izf(n+1)}-e^{izf(n)}|\leqslant 2|z|\,|f(n+1)-f(n)|,$$

we can deduce that $g(n,z)$ is completely multiplicative, i.e. $g(p^s,z)=g(p,z)^s$ for every z, p and s. By derivation in z, we have $f(p^s)=sf(p)$.

References

[1] P. Erdös: On the distribution function of additive functions. Ann. of Math., **47**, 1–20 (1946).

[2] I. Kátai: On a problem of P. Erdös. J. of Number Theory, **2**, 1–6 (1970).

[3] F. Skof: Sulle funzioni $f(n)$ arithmetiche additive asintotiche a $c(\log n)$. Ist Lombardo Accad. Sci. Lett. Rend. A, **103**, 931–938 (1969).