101. *On a Difference System of the Integrals of Pochhammer*

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In this note we investigate a difference system of the integrals of Pochhammer

(1) $$P_c(\hat{\lambda})=\int_c (\zeta-a_1)^{\lambda_1}\cdots(\zeta-a_n)^{\lambda_n}d\zeta,$$

with respect to the variable $\hat{\lambda}=(\lambda_1,\cdots,\lambda_n)$, for a suitable cycle C. As is well-known, all the functions $P_c(\hat{\lambda}+\hat{k})$ ($\hat{k}\in Z^n$) are expressed as linear combinations, with rational coefficients of $\hat{\lambda}$, in terms of $u_k(\hat{\lambda})=P_c(\hat{\lambda}-\hat{e}_k)$ $k=1,\cdots,n$, where $\hat{e}_k$ is the unit vector $(0,\cdots,\overset{k-th}{1},\cdots,0)$ (cf. [3, § 18.26]). The difference system is determined by the following

(2) $$u_i(\hat{\lambda}-\hat{e}_j)=(a_i-a_j)^{-1}(u_i(\hat{\lambda})-u_j(\hat{\lambda})) \qquad i\neq j,$$

with the fundamental relation

(3) $$\sum_{i=1}^{n}\lambda_i u_i(\hat{\lambda})=0.$$

The system (2) and (3) defines an element of a cocycle belonging to the cohomology $H^1(Z^n, GL_{n-1}(\mathbf{C}(\hat{\lambda})))$. But the structure of $H^1(Z^n, GL_{n-1}(\mathbf{C}(\hat{\lambda})))$ for $n\geqq 3$ seems generally very difficult to determine. Therefore, we consider the system of the following special type

(4) $$u_i(\hat{\lambda}-\hat{e}_k)=\sum_{j=1}^{n} b_{ij}^k u_j(\hat{\lambda}) \qquad i\neq k,$$

with the fundamental relation

(5) $$\sum_{i,j=1}^{n} a_{ij}\lambda_i u_j(\hat{\lambda})=0,$$

where a_{ij} and b_{ij}^k ($k=1,\cdots,n$) denote constant matrices of rank n and $n-1$, respectively.

Theorem 1 (A characterization of the Pochhammer system). *Suppose that the system* (4) *and* (5) *has* $(n-1)$ *linearly independent solutions which are meromorphic with respect to* $\hat{\lambda}$. *Then this system becomes* (2) *and* (3), *except for a constant multiple of each* $u_i(\hat{\lambda})$ $(i=1,\cdots,n)$.

From now on, we shall assume that $a_1,\cdots,a_n$ are real numbers such that $a_1<\cdots<a_n$. In the case of several variables, when we restrict ourselves to asymptotic expansions only in "rational directions", the solution of (2) is completely determined by a difference system of one variable ([1, Théorème 1.2]).

For nonzero vectors $\hat{k} \in \mathbf{Z}^n$, we have from (4) and (5),

$$u_i(\hat{\lambda}+\hat{k}) = \sum_{j=1}^{n} c_{ij}(\hat{k}\,;\hat{\lambda}) u_j(\hat{\lambda}), \tag{6}$$

where $c_{ij}(\hat{k}\,;\hat{\lambda}) \in \mathbf{C}(\hat{\lambda})$. By the method of R.D. Carmichael [2], we can obtain the asymptotic behavior of the solutions of (6).

Theorem 2. *For a generic direction* (ω) *defined by* $\hat{k}$, *the solutions* $u_i(\hat{\lambda}+t\hat{k})$ *of* (6) *have the following asymptotic expression as* $\hat{\lambda} \to +\infty$,

$$u_i(\hat{\lambda}+t\hat{k}) = \prod_{j=1}^{n} (\alpha - a_j)^{\lambda_j + tk_j} t^{r_i}(c_i + O(t^{-1})) \qquad r_i,\ c_i \in \mathbf{C}, \tag{7}$$

where the number α *satisfies the equation*

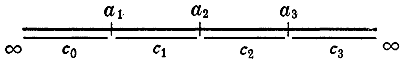
$$\sum_{i=1}^{n} k_i/(a_i - \alpha) = 0. \tag{8}$$

This is nothing else but the equation of the saddle points of the function $\mathrm{Re}\log \prod_1^n (\zeta - a_i)^{tk_i}$.

In the case of $n=3$, we want to calculate the connection matrices between the different rational directions. For this, by the "saddle point" method, we look for two cycles $C_1(\omega)$ and $C_2(\omega)$ of steepest descent, or homologous to them, giving two corresponding independent solutions $P_{c_1(\omega)}(\hat{\lambda})$ and $P_{c_2(\omega)}(\hat{\lambda})$ of (2) and (3).

In order to avoid all ambiguity of (1), we choose a branch of (1) as follows:

(i) For each cycle C_i $(i=0, \cdots, 3)$ to be figured below, (1) is $\exp -\pi i\lambda_1$ $[-\infty, a_1]$ ($[a_1, a_2]$, $\exp \pi i\lambda_2$ $[a_2, a_3]$, $\exp i(\lambda_2+\lambda_3)$ $[a_3, \infty]$), respectively.



(ii) For a closed cycle C, we take a branch of (1) such that the part of C below the real line is equal to some C_i $(i=0, \cdots, 3)$.

The coordinate ${}^t(P_{c_1}(\hat{\lambda}), P_{c_2}(\hat{\lambda}))$ is determined as follows:

(iii) For saddle points σ_i of C_i $(i=1, 2)$, if σ_1 is a real number, $\sigma_1 < \sigma_2$. Otherwise, $\mathrm{Im}\,\sigma_1 < \mathrm{Im}\,\sigma_2$.

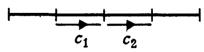
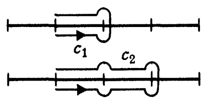
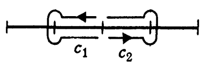
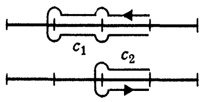
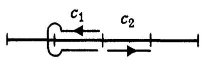
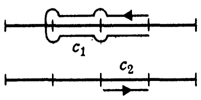
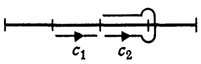
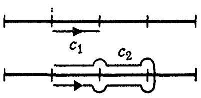
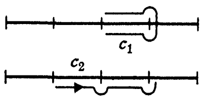
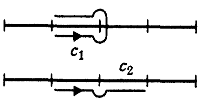
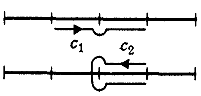
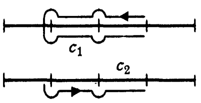
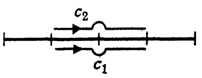
Remark. From (8), if $\mathrm{Im}\,\sigma_1 \neq 0$, σ_2 is the complex conjugate number of σ_1.

Under this situation, the connection matrix $E_{\omega\omega'}$ between two directions ω, ω' is defined by ${}^t(P_{c_1(\omega)}(\hat{\lambda}), P_{c_2(\omega)}(\hat{\lambda})) = E_{\omega\omega'}{}^t(P_{c_1(\omega')}(\hat{\lambda}), P_{c_2(\omega')}(\hat{\lambda}))$. We can calculate the connection matrices of (2) and (3).

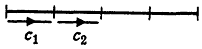
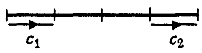
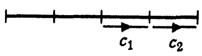
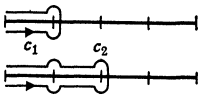
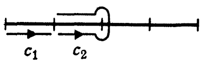
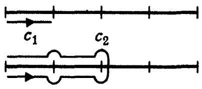
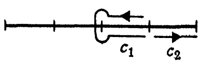
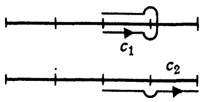
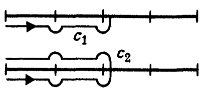
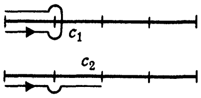
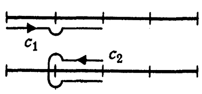
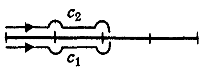
Theorem 3. *In the case of* $n=3$ *and* $a_1 < a_2 < a_3$, *there are, in general, certain hypersurfaces in the* (k_1, k_2, k_3)*-space beyond which the given asymptotic expansion of Theorem* 2 *alters, and their hypersurfaces give the following twenty six subdomains. Also the table of the connection matrices between the first domain*: $k_1>0$, $k_2>0$, $k_3>0$, *and the others are given below.*

Table of the connection matrices

(I) $k_1+k_2+k_3>0$,

subdomain of Z^3			cycles	connection matrix
$k_1>0,\ k_2>0,\ k_3>0$				
$k_1>0,\ k_2<0,\ k_3<0$				$\begin{pmatrix} 1-\varepsilon_2, & 0 \\ 1-\varepsilon_2\varepsilon_3, & 1-\varepsilon_3 \end{pmatrix}$
$k_1<0,\ k_2>0,\ k_3<0$				$\begin{pmatrix} 1-\varepsilon_1^{-1}, & 0 \\ 0, & 1-\varepsilon_3 \end{pmatrix}$
$k_1<0,\ k_2<0,\ k_3>0$				$\begin{pmatrix} 1-\varepsilon_1^{-1}, & 1-\varepsilon_1^{-1}\varepsilon_2^{-1} \\ 0, & 1-\varepsilon_2^{-1} \end{pmatrix}$
$k_1<0$, $k_2>0$, $k_3>0$		$k_1+k_2>0$		$\begin{pmatrix} 1-\varepsilon_1^{-1}, & 0 \\ 0, & 1 \end{pmatrix}$
		$k_1+k_2<0$		$\begin{pmatrix} 1-\varepsilon_1^{-1}, & 1-\varepsilon_1^{-1}\varepsilon_2^{-1} \\ 0, & 1 \end{pmatrix}$
$k_1>0$, $k_2>0$, $k_3<0$		$k_2+k_3>0$		$\begin{pmatrix} 1, & 0 \\ 0, & 1-\varepsilon_3 \end{pmatrix}$
		$k_2+k_3<0$		$\begin{pmatrix} 1, & 0 \\ 1-\varepsilon_2\varepsilon_3, & 1-\varepsilon_3 \end{pmatrix}$
$k_1>0$, $k_2<0$, $k_3>0$	$D>0$	(a)>0		$\begin{pmatrix} 0, & 1-\varepsilon_3 \\ 1, & 1 \end{pmatrix}$
		(a)<0, (b)>0		$\begin{pmatrix} 1-\varepsilon_2, & 0 \\ 1, & 1 \end{pmatrix}$
		(b)<0, (c)>0		$\begin{pmatrix} 1, & 1 \\ 0, & 1-\varepsilon_2^{-1} \end{pmatrix}$
		(c)<0		$\begin{pmatrix} 1-\varepsilon_1^{-1}, & 1-\varepsilon_1^{-1}\varepsilon_2^{-1} \\ 1, & 1 \end{pmatrix}$
	$D<0$			$\begin{pmatrix} 1, & 1 \\ \varepsilon_1^{-1}, & \varepsilon_1^{-1}\varepsilon_2^{-1} \end{pmatrix}$

(II) $k_1+k_2+k_3<0$

subdomain of Z^3			cycles	connection matrix
$k_1>0,\ k_2>0,\ k_3<0$				L_1
$k_1>0,\ k_2<0,\ k_3>0$				L_2
$k_1<0,\ k_2>0,\ k_3>0$				L_3
$k_1<0,\ k_2<0,\ k_3<0$				$\begin{pmatrix}1-\varepsilon_1, & 0\\ 1-\varepsilon_1\varepsilon_2, & 1-\varepsilon_2\end{pmatrix}L_1$
$k_1>0$ $k_2<0$ $k_3<0$	$k_1+k_2>0$			$\begin{pmatrix}1, & 0\\ 0, & 1-\varepsilon_2\end{pmatrix}L_1$
	$k_1+k_2<0$			$\begin{pmatrix}1, & 0\\ 1-\varepsilon_1\varepsilon_2, & 1-\varepsilon_2\end{pmatrix}L_1$
$k_1<0$ $k_2<0$ $k_3>0$	$k_2+k_3>0$			$\begin{pmatrix}1-\varepsilon_2^{-1}, & 0\\ 0, & 1\end{pmatrix}L_3$
	$k_2+k_3<0$			$\begin{pmatrix}1-\varepsilon_1, & 0\\ 0, & 1\end{pmatrix}L_2$
$k_1<0$ $k_2>0$ $k_3<0$	$D>0$	(a)<0		$\begin{pmatrix}1-\varepsilon_3, & 0\\ 1, & 1\end{pmatrix}L_3$
		(a)>0, (b)<0		$\begin{pmatrix}1, & 1\\ 1-\varepsilon_1\varepsilon_2, & 1\end{pmatrix}L_1$
		(b)>0, (c)<0		$\begin{pmatrix}1-\varepsilon_1, & 0\\ 1, & 1\end{pmatrix}L_1$
		(c)>0		$\begin{pmatrix}1, & 1\\ 0, & 1-\varepsilon_1^{-1}\end{pmatrix}L_1$
	$D<0$			$\begin{pmatrix}1, & 1\\ \varepsilon_1\varepsilon_2, & \varepsilon_2\end{pmatrix}L_1$

In this table the symbols denote the following:

$\varepsilon_j=\exp 2\pi i(\lambda_j+tk_j)j=1, 2, 3.$

$$L_1=(1-\varepsilon_1\varepsilon_2\varepsilon_3)^{-1}\begin{pmatrix}\varepsilon_2\varepsilon_3-1, & \varepsilon_3-1\\ 1-\varepsilon_1\varepsilon_2\varepsilon_3, & 0\end{pmatrix}.$$

$$L_2=(1-\varepsilon_1\varepsilon_2\varepsilon_3)^{-1}\begin{pmatrix}\varepsilon_2\varepsilon_3-1, & \varepsilon_3-1\\ \varepsilon_2\varepsilon_3(\varepsilon_1-1) & \varepsilon_3(\varepsilon_1\varepsilon_2-1)\end{pmatrix}.$$

$$L_3=(1-\varepsilon_1\varepsilon_2\varepsilon_3)^{-1}\begin{pmatrix}0, & 1-\varepsilon_1\varepsilon_2\varepsilon_3\\ \varepsilon_2\varepsilon_3(\varepsilon_1-1), & \varepsilon_3(\varepsilon_1\varepsilon_2-1)\end{pmatrix}.$$

$$(\mathrm{a})=(k_1+k_3)(a_2-a_3)+(k_2+k_3)(a_1-a_3).$$

$$(\mathrm{b})=(k_1+k_2)(a_3-a_2)+(k_2+k_3)(a_1-a_2).$$

$$(\mathrm{c})=(k_1+k_2)(a_3-a_1)+(k_1+k_3)(a_2-a_1).$$

D *is the discriminant of* (8).

The arrow —>— *means the direction of the cycles. The five points on the real line* ⊢—+—+—+—⊣ *denote* $-\infty$, a_1, a_2, a_3 *and* $+\infty$ *from the left to the right.*

Remark (K. Aomoto). From (7) and (8), $H^1(Z^3, GL_2(\mathbf{C}(\hat{\lambda})))$ contains $(1, +\infty)$ as a set. Thus the situation is much different from the case of $H^1(Z^3, GL_1(\mathbf{C}(\hat{\lambda})))$ (cf. [4]).

References

[1] Aomoto, K.: Les équations aux différences linéaires et les intégrales des fonctions multiformes. J. Fac. Sci. Univ. Tokyo, **22** (3), 271–297 (1975).

[2] Carmichael, R. D.: Linear difference equations and their analytic solutions. Trans. Amer. Math. Soc., **12**, 99–139 (1911).

[3] Ince, E. L.: Ordinary Differential Equations. Dover (1956).

[4] Sato, M.: Theory of prehomogeneous vector spaces (note by T. Shintani in Japanese), Sugakuno Ayumi, **15**, 85–157 (1970).

[5] Watson, G. N.: Asymptotic expansions of hypergeometric functions. Trans. Cambridge Philos. Soc., **22**, 277–308 (1918).