100. A Remark Concerning the Extensions of Some Group C*-Algebras

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We investigate the extensions of the enveloping group C^*-algebras of discrete groups and show that to the free product of groups corresponds the direct sum of EXTs. As a consequence, it will be seen that the EXT of the enveloping group C^*-algebra of a free group F_n is Z^n, a result announced in L. G. Brown [2].

Let $G_k(k \in N)$ be groups, then we denote by $G_1 * G_2$ (resp. $\prod^*_{k\in N} G_k$) the free product of G_1 and G_2 (resp. $\{G_k\}_{k\in N}$). If F is a group and φ_k is homomorphism of G_k into F, then there exists a unique homomorphism φ of $\prod^* G_k$ into F such that $\varphi \circ \iota_i = \varphi_i$ for all i, where ι_i is the canonical inclusion of G_i into $\prod^* G_k$. Throughout the paper, we assume that the groups are countable, $C^*(G)$ is then separable and has a unit, where $C^*(G)$ denotes the enveloping group C^*-algebra of G.

H is a separable infinite dimensional Hilbert space, $Q(H)$ is the Calkin algebra on H, and π is the quotient map from the total operator algebra $B(H)$ onto $Q(H)$. An extension τ of $K(H)$, the algebra of compact operators, by a unital separable C^*-algebra A is a unital *-isomorphism of A into $Q(H)$. $EXT(A)$ is the family of all equivalence classes of extensions by A. Concerning these, we follow mainly the expositions in [1].

Let φ be a unital *-homomorphism of A into another unital separable C^*-algebra B. φ induces a homomorphism $\varphi^\#$ of $EXT(B)$ into $EXT(A)$ in the following way. For $[\tau] \in EXT(B)$, $\varphi^\#[\tau] = [\tau \circ \varphi \oplus \tau_0]$, where τ_0 is the trivial extension of A, the extension which comes from a unital *-isomorphism of A into $B(H)$. This is well-defined because of the equivalence of all trivial extensions.

For short, we write $EXT[G]$ in place of $EXT(C^*(G))$.

Theorem. *Let G_k be discrete groups $(k \in N)$. If $EXT[G_k]$ are groups for all k, then $EXT[\prod^*_{k\in N} G_k]$ is a group. Moreover*

$$EXT[\textstyle\prod^*_{k\in N} G_k] = \prod_{k\in N} EXT[G_k].$$

Proof. If G is a discrete group, $C^*(G)$ is generated by $\{U_g ; g \in G\}$, where U_g is the corresponding unitary to $g \in G$ in its universal representation. The canonical injection ι_i of G_i into $\prod^* G_k$ induces a *-homomorphism $\iota_{i\#}$ of $C^*(G_i)$ into $C^*(\prod^* G_k)$. $\iota_{i\#}$ also induces a homomorphism $\iota_i^\#$ of $EXT[\prod^* G_k]$ into $EXT[G_i]$. We define a homomorphism Φ of $EXT[\prod^* G_k]$ into $\prod EXT[G_k]$ by

$$\Phi([\tau])=\iota_1^\#[\tau]\times\cdots\times\iota_n^\#[\tau]\cdots.$$

Surjection. For $[\tau_i]\in EXT[G_i]$ $(i\in N)$ there exists a homomorphism σ_i of G_i into the unitary group of $Q(H)$ corresponding to τ_i. Let p_i be a homomorphism of $\prod^* G_k$ into G_i such that $p_i\circ\iota_i=\mathrm{id}_{G_i}$, $p_i\circ\iota_j=e_{G_i}$ for $i\neq j$, e_{G_i} being the neutral element of G_i. Consider $Q(H)=Q(H\oplus H\oplus\cdots)$. We can then define a homomorphism σ of $\prod^* G_k$ into the unitary group of $Q(H)$ by $\oplus\sigma_i$. σ induces a *-homomorphism τ' of $C^*(\prod^* G_k)$ into $Q(H)$. Put now $\tau=\tau'\oplus\tau_0$, where τ_0 is a trivial extension of $C^*(\prod^* G_k)$. It is easy to see that $\iota_i^\#[\tau]=[\tau_i]$.

Injection. Because the range of Φ is a group, we have only to show that the kernel of Φ is trivial. If $\Phi[\tau]=0$, then $\iota_i^\#[\tau]=0$. Note that $p_{i\#}\circ\iota_{i\#}=(p_i\circ\iota_i)_\#=\mathrm{id}_{C^*(G_i)}$. $\iota_{i\#}$ is then an injection, so $\tau\circ\iota_i^\#$ is an extension. As it is trivial there is a unital *-homomorphism t_i of $C^*(G)$ into $B(H)$ such that $t_i=\tau\circ\iota_{i\#}$. We have a unitary representation s_i of G_i on H associated with t_i. By the universality of the free product, we have a unitary representation s of $\prod^* G_k$ with $s\circ\iota_i=s_i$ for all i. Then there exists a *-homomorphism t of $C^*(\prod^* G_k)$ into $B(H)$ associated with s. τ equals $\pi\circ t$ on the generators $\{U_g : g\in\bigcup\iota_k(G_k)\}$ of $C^*(\prod^* G_k)$. By continuity $\pi\circ t=\tau$, i.e. τ is a trivial extension.

Q.E.D.

Remark. Put $G_1=Z/2Z$, $G_2=Z/3Z$. Then $EXT[G_1*G_2]=0$ since $EXT[G_k]=0$. In general, when G is compact abelian, then $EXT[G]=0$ ([2]). Notice that the reduced group C^*-algebra of G_1*G_2 is the Choi-algebra ([4]), and $C^*(G_1*G_2)$ is an example of a non-nuclear C^*-algebra whose EXT is reduced to zero.

Corollary. *For $n=1,2\cdots,\infty$, $EXT[F_n]=Z^n$, where F_n is the free group with n-generators.*

Remark. When $n=2$, this is L. G. Brown's example announced in [2]. It is a non-nuclear C^*-algebra whose EXT is a group. We remark also that here we have a proof that the $C^*(F_n)$ are not isomorphic for different n.

If we use "direct sum" instead of "free product" in the statement of the theorem, the isomorphism does not always hold good. As a counter-example, we mention that $EXT[Z^3]=EXT(T^3)=Z^4$, where T^n is the n-dimensional torus and $EXT(T^3)$ is the extension group of the C^*-algebra $C(T^3)$ of all continuous functions on T^3. The first isomorphism follows from $C^*(G)=C(\hat{G})$ for the abelian group G, where $\hat{G}$ is a dual group of G. The second isomorphism is acquired by the following observations. First we cite two results from [3].

Theorem ([3, 7]). *Let B and C be closed subsets of a compact* *metrizable space X with $X=B\cup C$ and $A=B\cap C$. Then there is a cyclic exact sequence of six terms,*

$$\begin{array}{ccccc} EXT(A) & \longrightarrow & EXT(B)\oplus EXT(C) & \longrightarrow & EXT(X) \\ \uparrow & & & & \downarrow \\ EXT(SX) & \longleftarrow & EXT(SB)\oplus EXT(SC) & \longleftarrow & EXT(SA) \end{array}$$

where SX is suspension of X.

Lemma ([3, 5.1]). *If $p_1: X\times S^k\to X$, $p_2: X\times S^k\to S^k$ are projections, then there is a natural exact sequence*

$$0\to EXT(S^kX)\to \ker p_{1*}\to EXT(S^k)\to 0.$$

Applying the theorem in the above with $(T^2, I\times T^1, I\times T^1)$ in place of (X, B, C), where I denotes the interval $[0,1]$, we have

$$\begin{array}{ccccc} Z\oplus Z & \longrightarrow & Z\oplus Z & \longrightarrow & Z\oplus Z \\ \uparrow & & & & \downarrow \\ EXT(S(T^2)) & \longleftarrow & 0 & \longleftarrow & Z \end{array}$$

where $EXT(S(T\vee T))=Z$. In fact, if we use the theorem for the triple $(S(T\vee T), S^2, S^2)$, we have an exact sequence

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & EXT(S(T\vee T)) \\ \uparrow & & & & \downarrow \\ Z\oplus Z & \longleftarrow & Z\oplus Z & \longleftarrow & Z \end{array}$$

It follows that $EXT(S(T\vee T))=Z$. Hence, $EXT(S(T^2))=Z$. The lemma in the above implies, for $X=T^2$ $k=1$,

$$0\longrightarrow EXT(S(T^2))\longrightarrow \ker p_{1*}\xrightarrow[p_{2*}]{} EXT(T^1)\longrightarrow 0$$

hence $\ker p_{1*}=Z\oplus Z$.

We construct the following natural exact sequence

$$0\longrightarrow \ker p_{1*}\longrightarrow EXT(T^3)\xrightarrow[p_{1*}]{} EXT(T^2)\longrightarrow 0.$$

The exactness of the last part follows from

$$p_{1*}h_*=(p_1h)_*=(\mathrm{id}_{T^2})_*=\mathrm{id}_{EXT(T^2)}$$

where $h: T^2\to T^2\times(*)\to T^2\times T^1$ is the canonical injection.

From this we see that $EXT(T^3)=Z^4$.

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