98. *On a Result of T. Watanabe on Excessive Functions*

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Let $(\Omega, \mathcal{M}, \mathcal{M}_t, X_t, \theta_t, P^x)$ be a standard process with state space E (locally compact, denumerable base) and suppose that its resolvent $\{V_\lambda : \lambda > 0\}$ has the following property:

$$V_\lambda(C_b(E)) \subset C_b(E) \qquad \text{for each } \lambda > 0,$$

where $C_b(E)$ is the space of all bounded continuous functions on E.

The aim of this note is to prove the following result, which extends and unifies two results of T. Watanabe (Theorems 1 and 2 in [5]):

Theorem. *Let $f: E \to [0, \infty]$ be a lower semicontinuous function. Assume that for each $x \in E$ there exists a family of nearly Borel sets $\mathcal{U}(x)$ such that*

1° *$\mathcal{U}(x)$ is a base of neighbourhoods of x,*

2° *$E^x(f(X_{T_{CU}})) \leqslant f(x)$ for each $U \in \mathcal{U}(x)$.*

Then f is an excessive function.

The proof makes use of Bauer's minimum principle. We also need the following consequence of a result of G. Mokobodzki:

Lemma. *If the potential kernel V_o maps the space of all continuous functions with compact support $C_c(E)$ into $C_b(E)$, then for each $g \in C_{c+}(E)$,*

$$\inf\{t : t \text{ is a lower semicontinuous excessive function and } t \geqslant V_o g \text{ on } CK, \text{ for some compact set } K\} = 0$$

Proof. From Theorem 12, p. 231 of [3], we deduce for each lower semicontinuous function g, the function Rg defined by

$$Rg = \inf\{t : t \text{ is an excessive function and } t \geqslant g\}$$

is a lower semicontinuous excessive function. (It should be noted that in [3] are considered only Borel excessive functions but the methods work for universally measurable functions.) Therefore if $g \in C_c^+(E)$ and K is a compact set, then $R(\chi_{CK} V_o g)$ is lower semicontinuous. From Hunt's theorem (see [2], page 141) we know that $R(\chi_{CK} V_o g)(x) = E^x(V_o g(X_{T_{CK}})) = E^x\left(\int_{T_{CK}}^\infty g(X_t)dt\right)$ and hence $R(\chi_{CK} V_o g) \to 0$ when $K \nearrow E$, which implies the lemma.

Proof of the theorem. In order to simplify the exposition we first assume that the potential kernel V_o has also the property $V_o(C_b(E)) \subset C_b(E)$. Next we are going to prove $\lambda V_\lambda f \leqslant f$, for $\lambda > 0$. Since f

$=\sup\{h\in C_c^+(E): h\leqslant f\}$, we have only to prove $\lambda V_\lambda h\leqslant f$ for $h\in C_c^+(E)$, $h\leqslant f$. If we denote by $g_1=\max(0, h-\lambda V_\lambda h)$, $g_2=\max(0, \lambda V_\lambda h-h)$, the inequality becomes

$$f-\lambda V_\lambda h=f-\lambda V_o g_1+\lambda V_o g_2\geqslant 0.$$

By the lemma this inequality would follow if we proved

$$f-\lambda V_o g_1+t+\lambda V_o g_2\geqslant 0,$$

for each lower semicontinuous excessive function t which satisfy $\lambda V_o g_1 \leqslant t$ on CK, for some compact set K.

For t and K as above let us suppose that

(1) $$\alpha=\inf(f-\lambda V_o g_1+t+\lambda V_o g_2)<0.$$

Then the set

$$K_o=\{x\in E: f(x)-\lambda V_o g_1(x)+t(x)+\lambda V_o g_2(x)=\alpha\}$$

is compact, because K_o must satisfy $K_o\subset K$.

Now we are going to apply Bauer's minimum principle for the compact space K_o and the family $\mathcal{S}=\{t_{|K_o}: t$ is a lower semicontinuous excessive function$\}$. Since $\mathcal{S}$ separates points of K_o, from page 7 of [1], we get a point $y\in K_o$ such that each positive Radon measure μ on K_o which satisfy

a) $\mu(t)\leqslant t(y)$ for each $t\in\mathcal{S}$,

b) $\mu(1)=1$,

should coincide with εy.

Since $y\in K_o$ we have $h(y)-\lambda V_\lambda h(y)\leqslant f(y)-\lambda V_\lambda h(y)\leqslant f(y)-\lambda V_o g_1(y)+t(y)+\lambda V_o g_2(y)=\alpha<0$, and hence $y\notin \operatorname{supp} g_1$. Then we choose $U\in\mathcal{U}(y)$ such that $U\cap\operatorname{supp} g_1=\emptyset$. It follows

$$E^Y\left(\int_o^{T_{CU}} g_1(x_t)dt\right)=0,$$

which implies

$$E^y(V_o g_1(X_{T_{CU}}))=E^y\left(\int_{T_{CU}}^\infty g_1(X_t)dt\right)=E^y\left(\int_0^\infty g_1(X_t)dt\right)=V_o g_1(y).$$

Further from assumption 2° and the above relation we get

$$E^y((f-\lambda V_o g_1+t+\lambda V_o g_2)(X_{T_{CU}}))\leqslant f(y)-\lambda V_o g_1(y)+t(y)+\lambda V_o g_2(y)=\alpha.$$

Since $E^y(1(X_{T_{CU}}))\leqslant 1$, from relation (1) we get

$$\alpha\leqslant E^y(\alpha(X_{T_{CU}}))\leqslant E^y((f-\lambda V_o g_1+t+\lambda V_o g_2)(X_{T_{CU}}))\leqslant\alpha.$$

Therefore $E^y(X_{T_{CU}}\in K_o)=1$. We conclude that the measure μ defined by $\mu(f)=E^y(f(E_{T_{CU}}))$ for $f\in C_c(E)$ is a Radon measure supported by K_o and it satisfies relations a) and b).

On the other hand $X_{T_{CU}}\in\overline{CU}$, P^y—a.s. and $y\notin CU$. Thus $\mu\neq\varepsilon y$, which contradicts the asserted property of y. Finally our supposition fails, and hence $\alpha\geqslant 0$. It follows $\lambda V_\lambda f\leqslant f$.

Now let us consider the general case (where the potential kernel may be nonfinite). For $\alpha>0$ we first deduce

$$E^x(\exp(-\alpha T_{CU})f(X_{T_{CU}}))\leqslant E^x(f(X_{T_{CU}}))\leqslant f(x),$$

for each $U\in\mathcal{U}(x)$. Then from the first part of the proof, applied with respect to the α-subprocess, we have $\lambda V_{\lambda+\alpha} f \leqslant f$, for $\lambda > 0$. When $\alpha \to 0$ We get $\lambda V_\lambda f \leqslant f$.

It is well known that a nonnegative lower semicontinuous function which satisfies $\lambda V_\lambda f \leqslant f$, for each $\lambda > 0$, is excessive. Thus the theorem has been proved.

A preliminary version of this theorem is going to appear in [4].

References

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[5] T. Watanabe: On the Equivalence of Excessive Functions and Superharmonic Functions in the Theory of Markov Process. II. Proc. Japan Acad., **38**, 402–407 (1962).