97. *Approximate Innerness of Positive Linear Maps of Factors of Type II*

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We examine the properties of positive linear maps on approximately finite dimensional factors of type II that positive linear maps are approximately inner in a sense.

Arveson gets a useful property in [1] that a completely positive linear map ρ of a separable C^*-algebra A satisfying $\rho=0$ on $A \cap C(H)$ is approximately inner in a sense where A is acting on a Hilbert space H and $C(H)$ is the closed two-sided ideal of all compact operators on H.

In the present paper, we deal the approximately finite dimensional factor of type II (II_1 or II_∞) and show some properties of positive linear maps.

Let M be an approximately finite dimensional factor of type II_1 (resp. type II_∞) acting on a separable Hilbert space H. Let tr be a fixed, faithful, normal finite (resp. semifinite) trace of M. In the case of type II_1, we assume $tr(1)=1$.

Let F be the ideal generated by all finite projections in M and I the norm closure of F. Then, every projection in I is a finite projection. Furthermore, let $S=\{x \in M\,;\, x^*x \in F\}$, then S is an ideal in M and we can define a norm $\|\cdot\|_2$ on S by the following way; $\|x\|_2=tr(x^*x)^{1/2}$ for $x \in S$. If M is type II_1, then $F=S=I=M$.

Under the above notations, we can show that any strongly continuous positive linear map on an approximately finite dimensional factor of type II_1 is approximately inner with respect to the norm $\|\cdot\|_2$.

Theorem 1. *Let M be an approximately finite dimensional factor of type* II_1. *Let ρ be a strongly continuous positive linear map of M into M such that $\rho(1)=e$ is a projection, then there exists a sequence $\{\nu_n\}$ of partial isometries in M with the initial domain e such that*

$$\lim_{n\to\infty} \|\rho(x)-\nu_n^* x \nu_n\|_2=0$$

for every $x \in M$.

Proof. To prove the theorem, we show that, for an arbitrary positive number ε and finite set $\{a_1, \cdots, a_n\}$ of the unit ball of M, there exists a partial isometry ν with the initial domain e satisfying the following property;

$$(*) \qquad \|\nu^* a_i \nu-\rho(a_i)\|_2<\varepsilon \qquad \text{for } i=1, 2, \cdots, n.$$

Since the algebra eMe is an approximately finite dimensional factor of type II_1, for the set $\{\rho(a_1), \cdots, \rho(a_n)\}$, there exist a subfactor B of type I_{2^l} of eMe and a set $\{b_1, \cdots, b_n\}$ of elements in B satisfying

$$\|\rho(a_i)-b_i\|_2<\varepsilon/3 \qquad \text{for } i=1, 2, \cdots, n.$$

Let Φ be a normal expectation of eMe onto B such that $\mathrm{tr}\,(x)=\mathrm{tr}\,(\Phi(x))$ for every $x\in eMe$. Then, it is evident that Φ satisfies a property $\Phi(x)^*\Phi(x)\leqq\Phi(x^*x)$ for every $x\in eMe$,

Put $\rho'=\Phi\circ\rho$. To show the relation $(*)$, it is sufficient for us to show that there exists a partial isometry ν in M with the initial domain e satisfying;

$$(**) \qquad \|\nu^*a_i\nu-\rho'(a_i)\|_2<\varepsilon/3 \qquad \text{for } i=1, 2, \cdots, n.$$

In fact, suppose that there exists a partial isometry ν satisfying the property $(**)$. Then we have the following relation:

$$\begin{aligned}\|\rho'(a_i)-b_i\|_2&=\mathrm{tr}\,(((\Phi\circ\rho)(a_i)-\Phi(b_i))^*((\Phi\circ\rho)(a_i)-\Phi(b_i)))^{1/2}\\&=\mathrm{tr}\,(\Phi(\rho(a_i)-b_i)^*\Phi(\rho(a_i)-b_i))^{1/2}\\&\leqq\mathrm{tr}\,(\Phi((\rho(a_i)-b_i)^*(\rho(a_i)-b_i)))^{1/2}\\&=\mathrm{tr}\,((\rho(a_i)-b_i)^*(\rho(a_i)-b_i))^{1/2}=\|\rho(a_i)-b_i\|_2<\varepsilon/3\end{aligned}$$

for $i=1, 2, \cdots, n$. Thus, we have the following;

$$\begin{aligned}\|\rho(a_i)-\nu^*a_i\nu\|_2&\leqq\|\rho(a_i)-b_i\|_2+\|b_i-\rho'(a_i)\|_2+\|\rho'(a_i)-\nu^*a_i\nu\|_2\\&<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon\end{aligned}$$

for $i=1, 2, \cdots, n$. Hence, the relation $(**)$ implies the relation $(*)$.

Now, since ρ' is strongly continuous, for the above given positive number ε, there exists a positive number δ such that $\|\rho'(x)-\rho'(y)\|_2<\varepsilon/9$ for any pair x, y satisfying $\|x-y\|_2<\delta^2$. We can assume that δ is less than $\varepsilon/9\sqrt{2}$.

Since M is an approximately finite dimensional factor of type II_1, for δ, there exists a subfactor A of type I_{2^k} of M and a set $\{a_1', \cdots, a_n'\}$ in the unit ball of A such that $\|a_i-a_i'\|_2<\delta^2$ for every $i=1, 2, \cdots, n$. Let ν be an arbitrary partial isometry with the initial domain e, then we have the relation

$$\|\nu^*a_i\nu-\nu^*a_i'\nu\|_2<\varepsilon/9$$

for $i=1, 2, \cdots, n$.

Next, we consider the following property. If there exists a partial isometry ν in M with the initial domain e such that

$$(***) \qquad \|\nu^*a_i'\nu-\rho'(a_i')\|_2<\varepsilon/9$$

for $i=1, 2, \cdots, n$, then

$$\begin{aligned}&\|\nu^*a_i\nu-\rho'(a_i)\|_2\\&\quad\leqq\|\nu^*a_i\nu-\nu^*a_i'\nu\|_2+\|\nu^*a_i\nu-\rho'(a_i')\|_2+\|\rho'(a_i')-\rho'(a_i)\|_2\\&\quad<\varepsilon/9+\varepsilon/9+\varepsilon/9=\varepsilon/3\end{aligned}$$

for $i=1, 2, \cdots, n$. Thus, it is sufficient for us to show that there exists a partial isometry ν satisfying the relation $(***)$.

Since M is type II, we can assume that both A and B are factors of the same type I_{2^l}. Put $m=2^l$. Let $A=p_1\otimes B(C_A^m)$ and $B=e_1\otimes B(C_B^m)$

where C_A^m (resp. C_B^m) is an m-dimensional space with respect to the algebra A (resp. B) and $\sum_{i=1}^m p_i=1$, $\sum_{i=1}^m e_i=e$, $p_i\sim p_j$ and $e_i\sim e_j$ $(i, j=1, 2, \cdots, m)$. Then, ρ' is a unital positive linear map of A into B. Let π be the *-isomorphism of B onto $B(C_B^m)$ defined by $\pi(e_1\otimes c)=c$. Furthermore, let $\{\xi_i\}_{i=1}^m$ (resp. $\{\eta_i\}_{i=1}^m$) be a orthonormal basis of C_A^m (resp. C_B^m). Consider a positive linear functional φ on A defined by

$$\varphi(a)=\sum_{i,j=1}^m((\pi\circ\rho')(a)\eta_i|\eta_j).$$

Since M is type II, $M\cap C(H)=\{0\}$. Thus, by the Glimm's theorem [4], φ is represented as a limit of vector states. Furthermore, any vector state of A is represented as a limit of the following vector states 4ψ: That is, there exists a orthonormal system $\{\delta_i\}_{i=1}^s$ in p_1H and a set $\{\zeta_i\}_{i=1}^s$ in C_A^m such that

$$\psi(a)=((p_1\otimes c)(\sum_{i=1}^s\delta_i\otimes\zeta_i)|(\sum_{i=1}^s\delta_i\otimes\delta_i))$$

for every $a=p_1\otimes c\in A$. Put $\|\delta_i\|\zeta_i=\gamma_i$ for $i=1, 2, \cdots, s$, then $\psi(a)=\sum_{i,j=1}(c\gamma_i|\gamma_j)$ for $a=p_1\otimes c\in A$. For such ψ, there exists an operator x on C_A^m such that $\psi(a)=\sum_{i,j}^m(cx\xi_i|x\xi_j)$. Let y be a unitary operator of C_B^m onto C_B^m defined by $y\eta_i=\xi_i$ for $i=1, 2, \cdots, m$ and $w=xy$, then

$$\psi(a)=\sum_{i,j=1}^m(w^*cw\eta_i|\eta_j).$$

Thus, there exists a net $\{w_\iota\}$ of operators of C_B^m to C_A^m such that

$$\left|\sum_{i,j=1}^m(w_\iota^*cw_\iota\eta_i|\eta_j)-\sum_{i,j=1}^m((\pi\circ\rho')(a)\eta_i|\eta_j)\right|^2\longrightarrow 0$$

for every $a=p_1\otimes c\in A$. Since the space C_B^m is finite dimensional,

$$\|w_\iota^*cw_\iota-(\pi\circ\rho')(a)\|\longrightarrow 0$$

for every $a=p_1\otimes c\in A$. Furthermore, since $\rho'(1)=e$ (e is the identity of the algebra B), $w_\iota^*w_\iota$ is eventually invertible. For such 2, $x_\iota(w_\iota^*w_\iota)^{-1/2}$ is unitary operator and $(w_\iota^*w_\iota)^{-1/2}\to 1_m$ in the norm topology. Hence,

$$\|(w_\iota^*w_\iota)^{-1/2}w_\iota^*cw_\iota(w_\iota^*w_\iota)^{-1/2}-(\pi\circ\rho')(a)\|\longrightarrow 0.$$

From the above arguments, there exists a unitary operator w of C_B^m onto C_A^m such that

$$\|w^*c_iw-(\pi\circ\rho')(a_i')\|<\varepsilon/9\ \mathrm{tr}\,(e)^{1/2}$$

for $i=1, 2, \cdots, n$ where $a_i'=p_1\otimes c_i$ for $i=1, 2, \cdots, n$.

Let t be a partial isometry such that $t^*t=e_1$ and $tt^*=q_1\leqq p_1$. Let u_i be a partial isometry with the initial domain p_1 and the final domain p_i. Put $q_i=u_iq_1u_i^*$, then $(u_it)^*(u_it)=e_1$ and $(u_it)(u_it)^*=q_i$. Furthermore, let s_1 be a partial isometry with the initial domain e_1 and the final domain e_i. Put $u_{ij}=u_its_j^*$, then u_{ij} is a partial isometry with the initial domain e_j and the final domain q_i. Let $w=(w_{ij})$ be the matrix representation of w with respect to the orthonormal basis $\{\eta_j\}_{j=1}^m$ and $\{\xi_i\}_{i=1}^m$. From the above mentioned notations and some properties, put

$$\nu=\sum_{i,j=1}^m w_{ij}u_{ij}.$$

Then, we can show the following relation by an elementary computation;

$$\|\nu^* a_i \nu - \rho'(a_i')\| = \|w^* c_i w - (\pi \circ \rho')(a_i')\| < \varepsilon/9 \operatorname{tr}(e)^{1/2}.$$

Hence, there exists a partial isometry ν satisfying the relation (***). Therefore, we get the complete proof of Theorem 1.

By using the proof of Theorem 1, we can show a similar result for a C^*-subalgebra of M without the assumption of strong continuity.

Proposition 2. *Let M be an approximately finite dimensional factor of type II_1 and ρ a bounded positive linear map of M into M such that $\rho(1)=e$ is a projection. Then, for an arbitrary C^*-subalgebra A of M contained by AF-C^*-subalgebra, there exists a sequence $\{\nu_n\}$ of partial isometries in M with the initial domain e satisfying*

$$\lim_{n\to\infty} \left\| \rho(x) - \nu_n^* x \nu_n \right\|_2 = 0$$

for every $x \in A$.

Next, we shall consider the above properties in the case of type II_∞. Then, we use the ideal I generated by all finite projections. In this case, we can get a similar result to Proposition 2 and a generalization of Arveson's result [1]. To get this result, we use the notation of MAF-subalgebra of M introduced in [5]. That is, a C^*-subalgebra A of M is called by an MAF-subalgebra of M if there exists a sequence $\{A_k\}$ of finite dimensional C^*-subalgebra of M with the identity p_k satisfying $A \subset \overline{\cup\{A_k + I\}}$ and furthermore the following properties: (1) Every non zero central projection of each A_k is infinite projection, (2) $1-p_k$ are finite projections ($k=1, 2, \cdots$), (3) $A_k + I \subset A_{k+1} + I$ ($k=1, 2, \cdots$). In [5, Theorem 4], we showed the similar result to Proposition 2 for a completely positive linear map on an arbitrary MAF-subalgebra. But, in the present paper, we shall show this property for a positive linear map. We can get the proof by using a similar way in Theorem 1.

Theorem 3. *Let M be an approximately finite dimensional factor of type II_∞ and e a finite projection in M. Let ρ be a positive linear map of M into eMe such that $\rho(1)=e$ and $\rho=0$ on I. Then, for an arbitrary, separable MAF-subalgebra A of M, there exists a sequence $\{\nu_n\}$ of partial isometries with the initial domain e such that*

$$\lim_{n\to\infty} \|\nu_n^* x \nu_n - \rho(x)\|_2 = 0$$

for every $x \in A$.

Corollary 4. *In Theorem 3, if $\rho(1)=1$, then there exists an isometry ν in M such that $\rho(x) - \nu^* x \nu \in I$ for every $x \in A$.*

If we consider the completely positive linear maps, then we can drop down the assumption that ρ is defined on M. Because, both algebras M and eMe are injective ([2] and [3]).

Added in proof (November 15, 1980). This article was done while

the author was staying at the University of Copenhagen in Denmark. After this article was sent for the publication from Copenhagen, the author and George A. Elliott made a joint work titled by "On the extensions of C^*-algebras relative to factors of type II_∞" in which the results of [5] in References are included.

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