96. *Calculus on Gaussian White Noise. II*

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We are going to reformulate the works of Hida [1], [2] to establish a calculus on generalized Brownian functionals which we call Hida calculus.

In Part I [11], we have prepared fundamental tools. By using them, we will discuss on generalized random variables, annihilation operators ∂_t, creation operators ∂_t^*, multiplications $x(t)\cdot$ and so forth.

§5. Generalized random variables. As assumed in §4 of Part I [11], let T be a separable metrizable space with a σ-finite Borel measure ν and put $E_0=L^2(T,\nu)$. Let $\mathcal{E}$ be a dense subset of E_0 which has a consistent sequence of inner products $\{(\xi,\eta)_p\,;\,p\geq 0\}$ such that

(5.1) $\qquad (\xi,\xi)_p\leq\rho(\xi,\xi)_{p+1},\quad$ for $p\geq 0$ with $\rho,\ 0<\rho<1$.

Let E_p be the completion of $\mathcal{E}$ by the norm $\|\ \|_p$ and $E_{-p}=E_p^*$ with $(\xi,\eta)_{-p}$ be the dual of E_p. Suppose that $\mathcal{E}$ is identical to the projective limit E_∞ of E_p. Then the dual $\mathcal{E}^*$ is the inductive limit $E_{-\infty}$ of E_{-p}. Throughout this note we assume that the injection $\iota_{0,1}$ from E_1 to E_0 is *traceable*; that is, $\delta_t:\xi\mapsto\xi(t)$ belongs to E_{-1} and the mapping $t\in T\to\delta_t\in E_{-1}$ is continuous, and assume that $\|\delta\|^2\equiv\int_T\|\delta_t\|_{-1}^2\,d\nu(t)<\infty$. Then by Lemma 4.2, the injection $\iota_{0,1}$ is a Hilbert-Schmidt operator. Therefore, by Gelfand-Minlos-Sazanov's theorem, we have

Theorem 5.1. *There exists a probability measure μ on $\mathcal{E}^*$ such that*

$$\int_{\mathcal{E}^*}e^{i\langle x,\xi\rangle}d\mu(x)=\exp\left[-\frac{1}{2}\|\xi\|_0^2\right],\qquad \textit{for } \xi\in\mathcal{E}.$$

Definition 5.2. The measure μ on $\mathcal{E}^*$ is called a *measure of Gaussian white noise*. The L^2-space $L^2(\mathcal{E}^*,\mu)$ is denoted by (L^2), simply.

It is well known that the measure μ is quasi-invariant under the shift $x\to x-\xi$ for $\xi\in\mathcal{E}$ and that

$$\text{(5.2)}\qquad \frac{d\mu(x-\xi)}{d\mu(x)}=\exp\left[\langle x,\xi\rangle-\frac{1}{2}\|\xi\|_0^2\right]\in L^q(\mathcal{E}^*,\mu)$$

for $q\geq 1$ [7]. With the result, we can define a transformation $\mathcal{S}$ by

$$\text{(5.3)}\qquad (\mathcal{S}\varphi)(\xi)=\int_{\mathcal{E}^*}\varphi(x+\xi)d\mu(x),\quad \xi\in\mathcal{E},\quad \varphi\in L^q(\mathcal{E}^*,\mu),\quad 1<q<\infty.$$

Remark 5.3. By (5.2) and (5.3), $(\mathcal{S}\varphi)(\lambda\xi)$ can be extended to an entire function of λ as follows;

$$(5.4)\qquad (\mathcal{S}\varphi)(\lambda\xi)=\int_{\mathcal{E}^*}\varphi(x)\exp\left[\lambda\langle x,\xi\rangle-\frac{\lambda^2}{2}\|\xi\|_0^2\right]d\mu,\qquad \xi\in\mathcal{E}.$$

Hence, the analytic continuation $(\mathcal{S}\varphi)(i\xi)$ satisfies

$$(5.5)\qquad (\mathcal{S}\varphi)(i\xi)=(\mathcal{T}\varphi)(\xi)\exp\left[\frac{1}{2}\|\xi\|_0^2\right],$$

where $\mathcal{T}$ is the transformation introduced by Hida-Ikeda [5];

$$(5.6)\qquad (\mathcal{T}\varphi)(\xi)=\int_{\mathcal{E}^*}e^{i\langle x,\xi\rangle}\varphi(x)d\mu(x).$$

Let $\mathcal{F}^{(p)}$ be the Hilbert space of functionals of $\xi\in\mathcal{E}$ spanned by $\{e^{\langle\eta,\xi\rangle};\ \eta\in\mathcal{E}\}$ (see § 3 of Part I) with inner product

$$(5.7)\qquad (e^{\langle\eta,\xi\rangle},e^{\langle\zeta,\xi\rangle})^{(p)}=\exp[(\eta,\zeta)_p].$$

Theorem 5.4. *The space* (L^2) *is isomorphic to* $\mathcal{F}^{(0)}$ *by* $\mathcal{S}$.

By (3.2) of Part I, $\mathcal{F}^{(p+1)}\subset\mathcal{F}^{(p)}\subset\mathcal{F}^{(0)}$ for $p\geq 1$. Put $\mathcal{H}^{(p)}=\mathcal{S}^{-1}(\mathcal{F}^{(p)})$ for $p\geq 0$, and induce inner product $(\ ,\ )_{\mathcal{H}^{(p)}}$ on $\mathcal{H}^{(p)}$ from the inner product of $\mathcal{F}^{(p)}$. Let $\mathcal{H}^{(-p)}$ be the dual of $\mathcal{H}^{(p)}$, $p>0$. Then we have inclusions.

$$(5.8)\qquad \begin{aligned}\mathcal{H}&=\mathcal{H}^{(\infty)}\subset\cdots\subset\mathcal{H}^{(p)}\subset\cdots\subset\mathcal{H}^{(0)}\\&=(L^2)\subset\cdots\subset\mathcal{H}^{(-p)}\subset\cdots\subset\mathcal{H}^{(-\infty)}=\mathcal{H}^*.\end{aligned}$$

Definition 5.5. We say that an element of $\mathcal{H}^*$ is a *generalized random variable* and that $\mathcal{H}$ is the space of *testing random variables.*

Lemma 5.6. (i) *$\{\varphi_n\}$ in $\mathcal{H}^{(p)}$ converges to φ weakly, if and only if it is bounded in $\mathcal{H}^{(p)}$ and $(\mathcal{S}\varphi_n)(\xi)$ converges to $(\mathcal{S}\varphi)(\xi)$ for each $\xi\in\mathcal{E}$.*

(ii) *If $\{\varphi_n\}$ is bounded in $\mathcal{H}^{(p)}$, $p\geq 1$ (or $p=0$), and if $(\mathcal{S}\varphi_n)(\xi)$ converges for each $\xi\in\mathcal{E}$, then it converges strongly in $(L^2)=\mathcal{H}^{(0)}$ (or in $\mathcal{H}^{(-1)}$, respectively).*

Lemma 5.7. *Suppose that $\mathcal{E}$ is a nuclear space. Then*

(i) *$\{\varphi_n\}$ in $\mathcal{H}$ converges strongly in $\mathcal{H}$, if and only if it is bounded in $\mathcal{H}$ and $(\mathcal{S}\varphi_n)(\xi)$ converges for each $\xi\in\mathcal{E}$,*

(ii) *the same assertion holds in $\mathcal{H}^*$.*

The Hermite polynomials with parameter α are defined by the generating function

$$(5.9)\qquad \exp\left[tu-\frac{\alpha}{2}t^2\right]=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(u;\alpha).$$

Remark 5.8. Our Hermite polynomial $H_n(u;\alpha)$ is equal to Kakutani's one up to $n!$ [8], in particular

$$H_n(u;0)=u^n,\qquad H_0(u;\alpha)=1,$$

$$H_{2n}(0;\alpha)=\frac{(2n)!}{n!2^n}(-\alpha)^n\quad\text{and}\quad H_{2n+1}(0;\alpha)=0.$$

Lemma 5.9. *We have the following formulae*

$$\mathcal{S}(H_n(\langle\cdot,\eta\rangle;\alpha)e^{\langle\cdot,\xi\rangle})(\xi)=H_n((\zeta+\xi,\eta)_0;\alpha-\|\eta\|_0^2)e^{(\xi,\zeta)_0+\|\zeta\|_0^2/2},$$

$$\mathcal{S}(H_n(\langle\cdot,\eta\rangle;\|\eta\|_0^2))(\xi)=(\xi,\eta)_0^n.$$

§ 6. Derivatives and their duals. Let φ be in $\mathcal{H}$, then $(\mathcal{S}\varphi)(\xi)$ is in $\mathcal{F}$ by definition. By Theorem 4.4 of Part I, the functional derivative $\delta/\delta\xi(t)$ is a continuous operator on $\mathcal{F}$. Therefore we can define a continuous operator $\partial/\partial x(t)$ on $\mathcal{H}$ by

$$\frac{\partial}{\partial x(t)}\varphi=\mathcal{S}^{-1}\frac{\delta}{\delta\xi(t)}(\mathcal{S}\varphi)(\xi). \tag{6.1}$$

Theorem 6.1. (i) *The operator $\partial/\partial x(t)$ is continuous on $\mathcal{H}$ and strongly continuous in t and satisfies*

$$\Big(\mathcal{S}\frac{\partial}{\partial x(t)}\varphi\Big)(\xi)=(\mathcal{S}\varphi)^{(1)}(\xi\,;t) \qquad \text{for } \varphi\in\mathcal{H},$$

$$\Big\|\frac{\partial}{\partial x(t)}\varphi\Big\|_{\mathcal{H}^{(p)}}\le\|\delta_t\|_{-1}\,\|\varphi\|_{\mathcal{H}^{(p+1)}}\,\rho^p(1-\rho^2)^{-1}.$$

(ii) *The dual operator $(\partial/\partial x(t))^*$ is continuous on $\mathcal{H}^*$ and strongly continuous in t and*

$$\Big(\mathcal{S}\Big(\frac{\partial}{\partial x(t)}\Big)^*\Psi\Big)(\xi)=\xi(t)(\mathcal{S}\Psi)(\xi) \quad \text{for } \Psi\in\mathcal{H}^* \quad\text{and}\quad \xi\in\mathcal{E},$$

$$\Big\|\Big(\frac{\partial}{\partial x(t)}\Big)^*\Psi\Big\|_{\mathcal{H}^{(-p)}}\le\|\delta_t\|_{-1}\,\|\Psi\|_{\mathcal{H}^{(-p+1)}}\,\rho^{p-1}(1-\rho^2)^{-1}.$$

For simplicity, denote

$$\partial_t=\frac{\partial}{\partial x(t)} \quad\text{and}\quad \partial_t^*=\Big(\frac{\partial}{\partial x(t)}\Big)^*. \tag{6.2}$$

By Theorem 6.1, we can define operators $A(f)$ on $\mathcal{H}$ and $A^*(f)$ on $\mathcal{H}^*$ by

$$A(f)\equiv\int_{T^m}d\nu(t_1)\cdots d\nu(t_m)f(t_1,\cdots,t_m)\partial_{t_1}\cdots\partial_{t_m},$$

$$A^*(f)\equiv\int_{T^m}d\nu(t_1)\cdots d\nu(t_m)f(t_1,\cdots,t_m)\partial_{t_1}^*\cdots\partial_{t_m}^*,$$

for f in $E_0^{\hat{\otimes}m}=\hat{L}^2(T^m,d\nu^m)$.

Theorem 6.2. *For $\varphi\in\mathcal{H}$, $\Psi\in\mathcal{H}^*$ and $f\in\mathcal{E}^{\hat{\otimes}m}$, we have*

(i) $(\mathcal{S}(A(f)\varphi))(\xi)=\langle(\mathcal{S}\varphi)^{(m)}(\xi\,;\cdot),f\rangle$,

$\|A(f)\varphi\|_{\mathcal{H}^{(p)}}\le\|f\|_{E_{-p}^{\hat{\otimes}m}}\,\|\varphi\|_{\mathcal{H}^{(p+1)}}\,(1-\rho^2)^{-(m+1)/2}\rho^m\sqrt{m!}$.

(ii) $(\mathcal{S}(A^*(f)\varphi))(\xi)=\langle f,\xi^{\hat{\otimes}m}\rangle(\mathcal{S}\varphi)(\xi)$,

$\|A^*(f)\varphi\|_{\mathcal{H}^{(p)}}\le\|f\|_{E_p^{\hat{\otimes}m}}\,\|\varphi\|_{\mathcal{H}^{(p+1)}}\,(1-\rho^2)^{-(m+1)/2}\sqrt{m!}$.

(iii) $\langle\Psi,A(f)\varphi\rangle=\langle A^*(f)\Psi,\varphi\rangle$ and $\langle A(f)\Psi,\varphi\rangle=\langle\Psi,A^*(f)\varphi\rangle$.

(iv) $A(f)A(g)=A(f\hat{\otimes}g)$ and $A^*(f)A^*(g)=A^*(f\hat{\otimes}g)$,

$A(f)A^*(g)-A^*(g)A(f)=(f,g)_0, \qquad$ if $f,g\in\mathcal{E}$.

Remark 6.3. By this theorem, $A(f)$, for $f\in E^{\hat{\otimes}m}$, can be regarded as continuous operators on both spaces $\mathcal{H}$ and $\mathcal{H}^*$. Further, $A(F)$, for $F\in E^{*\hat{\otimes}m}$, can be defined as a continuous operator on $\mathcal{H}$ while $A^*(F)$ is defined as a continuous operator on $\mathcal{H}^*$. In particular for F in $E_0^{\hat{\otimes}m}=L^2(T^m,\nu^m)$, $A^*(F)1$ is in (L^2).

By the theorem together with Theorems 3.1 and 4.4, we have

Lemma 6.4. *Let f be in $E^{\hat{\otimes}m}$ and put $\varphi=A^*(f)1$. Then*

$(\mathcal{S}\varphi)(\xi)=\langle f,\xi^{\hat{\otimes}m}\rangle$ *and* $\|\varphi\|_{\mathcal{H}^{(p)}}^2=\|\langle f,\xi^{\hat{\otimes}m}\rangle\|_{\mathcal{F}^{(p)}}^2=m!\,\|f\|_{E_p^{\hat{\otimes}m}}^2$

hold. Furthermore for $m>k$,

$$\partial_{t_1}\cdots\partial_{t_k}\varphi=\frac{m!}{(m-k)!}A^*(\delta^*_{t_1}\cdots\delta^*_{t_k}f)1.$$

Theorem 6.5. *Let φ be in $\mathcal{H}$, then*

$$\varphi=\sum_{k=0}^{\infty}\frac{1}{k!}A^*((\mathcal{S}\varphi)^{(k)}(0\,;\,\cdot))1$$

and

$$\|\varphi\|^2_{(L^2)}=\sum_{k=0}^{\infty}\frac{1}{k!}\int_{T^k}|(\mathcal{S}\varphi)^{(k)}(0\,;t_1,\cdots,t_k)|^2\,d\nu(t_1)\cdots d\nu(t_k).$$

Remark 6.6. As in Remark 4.5, $\partial_{t_1}\cdots\partial_{t_k}$ can be regarded as an operator-valued —from $\mathcal{H}^{(-p)}$ to $\mathcal{H}^{(-p-1)}$— generalized function.

§ 7. Multiplication and normal ordering. By Theorem 6.3, the operators ∂_t and ∂^*_t can be regarded as operator-valued generalized functions on $\mathcal{E}$. The commutation relations (iv) in Theorem 6.2 can be written in the following more symbolical forms;

$$\begin{aligned}\partial_t\partial^*_s-\partial^*_s\partial_t&=\delta_s(t),\\ \partial_t\partial_s-\partial_s\partial_t&=\partial^*_t\partial^*_s-\partial^*_s\partial^*_t=0.\end{aligned}\tag{7.1}$$

The relations are so-called the *canonical commutation relations*. According to the terminology in quantum field theory, ∂^*_t is called a *creation operator* and ∂_t is an *annihilation operator* at t.

Remark 7.1. Since $\varphi(x)$ and $\psi(x)$ in $\mathcal{H}$ are random variables in (L^2), the product $(\varphi\psi)(x)=\varphi(x)\psi(x)$ is a random variable, at least belonging to $L^1(\mathcal{E}^*,\mu)$. Later we will see that $\varphi\psi$ is in $\mathcal{H}$.

Theorem 7.2. *Define $x(t)\cdot\equiv\partial_t+\partial^*_t$, then for $\varphi\in\mathcal{H}$, $\eta\in\mathcal{E}$,*

$$\langle x,\eta\rangle\varphi=\int_T d\nu(t)\eta(t)x(t)\cdot\varphi=(A(\eta)+A^*(\eta))\varphi,$$

$$x(t)\cdot\varphi=A^*(n\delta^*_t f_n)1+A^*(\delta_t\hat{\otimes}f_n)1,\qquad \text{for } \varphi=A^*(f_n)1.$$

Let us use the notation of the *normal ordering* $:P:$ for polynomials P of ∂_t and ∂^*_t's (see [9], [10]). Then the following lemma is useful.

Lemma 7.3.

(i) $$:x(t_1)\cdots x(t_n)\cdot:=\sum_{\varDelta\subset\{1,\cdots,n\}}\prod_{j\in\varDelta}\partial^*_{t_j}\prod_{i\in\{1,\cdots,n\}\setminus\varDelta}\partial_{t_i},$$

$$:x(t_1)\cdots x(t_n)\cdot:1=\partial^*_{t_1}\cdots\partial^*_{t_n}1,$$

(ii) $$x(t_1)\cdots x(t_n)\cdot 1=\sum_{k=0}^{[n/2]}\sum_{\varDelta_1+\cdots+\varDelta_k+\varDelta_0=\{1,\cdots,n\}}\delta_{\varDelta_1}\cdots\delta_{\varDelta_k}\prod_{j\in\varDelta_0}\partial^*_{t_j},$$

where $\delta_\varDelta=\delta_{t_k}(t_m)$ if $\varDelta=\{k,m\}$.

Define a mapping from $\mathcal{E}^{\hat{\otimes}n}\times\mathcal{E}^{\hat{\otimes}m}$ into $\mathcal{E}^{\hat{\otimes}(n+m-2k)}$ for $0\le k\le n\wedge m\equiv\min\{n,m\}$ by

$$\begin{aligned}&f\otimes_{(k)}g(u_1,\cdots,u_{n+m-2k})\\ &=\frac{1}{(n+m-2k)!}\sum_{\sigma\in\mathfrak{S}_{n+m-2k}}\int_{T^k}f(u_{\sigma(1)},\cdots,u_{\sigma(n-k)},v_1,\cdots,v_k)\\ &\qquad\times g(u_{\sigma(n-k+1)},\cdots,u_{\sigma(n+m-2k)},v_1,\cdots,v_k)d\nu^k(v),\end{aligned}\tag{7.3}$$

here $\mathfrak{S}_{n+m-2k}$ is the symmetric group of order $(n+m-2k)$.

Theorem 7.4. *Let f be in $\mathcal{E}^{\hat{\otimes}m}$ and g be in $\mathcal{E}^{\hat{\otimes}n}$, then*

(i) $\|f\otimes_{(k)}g\|_{E_p^{\hat{\otimes}(n+m-2k)}}\leq\|f\|_{E_p^{\hat{\otimes}m}}\|g\|_{E_p^{\hat{\otimes}n}}\rho^{2kp}$,

(ii) *put* $\varphi(x)=A^*(f)\cdot 1$ *and* $\psi(x)=A^*(g)\cdot 1$, *then*

$$\varphi(x)\psi(x)=\sum_{k=0}^{n\wedge m}\frac{n!\,m!}{k!\,(n-k)!\,(m-k)!}A^*(f\otimes_{(k)}g)\cdot 1$$
$$=\int_{T^m}d\nu(t_1)\cdots d\nu(t_n)g(t_1,\cdots,t_n):x(t_1)\cdots x(t_n)\cdot:\varphi,$$

(iii) $$(\mathcal{S}(\varphi\psi))(\xi)=\sum_{k=0}^{n\wedge m}\frac{n!\,m!}{k!\,(n-k)!\,(m-k)!}\langle f\otimes_{(k)}g,\xi^{\hat{\otimes}(n+m-2k)}\rangle$$
$$=\sum_{k=0}^{n\wedge m}\frac{1}{k!}\int_{T^k}(\mathcal{S}\varphi)^{(k)}(\xi;t_1,\cdots,t_k)\times(\mathcal{S}\varphi)^{(k)}(\xi;t_1,\cdots,t_k)d\nu^k(t),$$

(iv) $$\|\varphi\psi\|_{\mathcal{H}^{(p)}}\leq 2^{n+m}\sum_{k=0}^{n\wedge m}\left(\frac{\|\delta\|^2\rho^{2p-2}}{2}\right)^k\|\varphi\|_{\mathcal{H}^{(p)}}\|\psi\|_{\mathcal{H}^{(p)}}.$$

Theorem 7.5. *Let φ and ψ be in $\mathcal{H}$, then $\varphi\psi$ belongs to $\mathcal{H}$ and*

(i) $\|\varphi\psi\|_{\mathcal{H}^{(p)}}\leq 5\,\|\varphi\|_{\mathcal{H}^{(p+q)}}\,\|\psi\|_{\mathcal{H}^{(p+q)}}$

holds for sufficiently large q such that $(4+\|\delta\|^2)\rho^q<1$,

(ii) $$(\mathcal{S}(\varphi\psi))(\xi)=\sum_{k=0}^{\infty}\frac{1}{k!}((\mathcal{S}\varphi)^{(k)}(\xi;\cdot),(\mathcal{S}\psi)^{(k)}(\xi;\cdot))_{E_0^{\hat{\otimes}k}}.$$

Theorem 7.6. *The multiplication operator $\varphi\cdot:\psi\to\varphi\psi$ is continuous and symmetric on $\mathcal{H}$. For φ and $\psi\in\mathcal{H}$, we have*

$$\partial_t(\varphi\psi)=\varphi\partial_t\psi+\psi\partial_t\varphi,$$
$$\partial_t^*(\varphi\psi)=\varphi\cdot\partial_t^*-(\partial_t\varphi)\cdot\psi.$$

Let $U(\xi)$ be in $\mathcal{F}$, then U can be extended to a continuous $\mathcal{H}$-functional $U(x)$ on $\mathcal{E}^*$. By Theorem 3.1, there exists a $\Xi=(f_0,\cdots,f_n,\cdots)\in e^{\hat{\otimes}\mathcal{E}}$ such that

$$U(x)=\sum_{n=0}^{\infty}\langle x^{\hat{\otimes}n},f_n\rangle. \tag{7.4}$$

The multiplication by $U(x)$ coincides with the operator

$$U(x)\cdot=\sum_{n=0}^{\infty}\langle x^{\hat{\otimes}n},f_n\rangle\cdot, \tag{7.5}$$

and its normal ordering is given by

$$:U(x)\cdot:=\sum_{n=0}^{\infty}:\langle x^{\hat{\otimes}n},f_n\rangle\cdot:=\sum_{n=0}^{\infty}A^*(f_n). \tag{7.6}$$

Therefore, we have

Theorem 7.7. *If $U(\xi)$ is in $\mathcal{F}$, then U can be extended to a continuous functional $U(x)$ on $\mathcal{E}^*$. Furthermore $U(x)$ is in $\mathcal{H}$ and satisfies*

$$\mathcal{S}(:U(x)\cdot:1)(\xi)=U(\xi).$$

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