95. *Studies on Holonomic Quantum Fields. XVII*

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We report the following two results on the diagonal spin-spin correlation function $\langle\sigma_{00}\sigma_{NN}\rangle$ of the two dimensional Ising lattice. (i) $\langle\sigma_{00}\sigma_{NN}\rangle$ satisfies a non-linear ordinary differential equation with respect to the temperature, which is equivalent to a sixth Painlevé equation (P VI). (ii) $\langle\sigma_{00}\sigma_{NN}\rangle$ satisfies a non-linear ordinary difference equation with respect to N. In the scaling limit, both the differential equation (i) and the difference equation (ii) reduce to known results [1] related to P V (or an equivalent of it, P III [2]) on the scaled two point function.

Our method is to construct an isomonodromy family of linear differential equations in such a way that its τ function [3] coincides with $\langle\sigma_{00}\sigma_{NN}\rangle$. The difference equation (ii) is a consequence of the relations among the τ function and its Schlesinger transforms [4], [5].

Recently McCoy-Wu [6] and Perk [7] have obtained difference equations for $\langle\sigma_{00}\sigma_{MN}\rangle$. The relations between their works and ours (for $M=N$) is yet to be clarified.

1. Results. We follow the notations of [8], [9]. Let $\langle\sigma_{00}\sigma_{NN}\rangle_{T_-<T_c}$ (resp. $\langle\sigma_{00}\sigma_{NN}\rangle_{T_+>T_c}$) denote the diagonal spin-spin correlation function below (resp. above) the critical temperature, where we use the parametrization

(1) $$t=(\sinh\beta_-E_1\sinh\beta_-E_2)^2=(\sinh\beta_+E_1\sinh\beta_+E_2)^{-2}$$

with $t>1$, $\beta_\pm=1/kT_\pm$. We set

(2) $$\sigma_{N,-}(t)=t(t-1)\frac{d}{dt}\log\langle\sigma_{00}\sigma_{NN}\rangle_{T_-<T_c}-\frac{1}{4},$$

$$\sigma_{N,+}(t)=t(t-1)\frac{d}{dt}\log\langle\sigma_{00}\sigma_{NN}\rangle_{T_+>T_c}-\frac{1}{4}t.$$

Then both $\sigma=\sigma_{N,\pm}(t)$ are solutions of the following second order non-linear ordinary differential equation.

(3) $$\left(t(t-1)\frac{d^2\sigma}{dt^2}\right)^2$$

$$=N^2\left((t-1)\frac{d\sigma}{dt}-\sigma\right)^2-\frac{d\sigma}{dt}\left((t-1)\frac{d\sigma}{dt}-\sigma-\frac{1}{2}\right)\left((t+1)\frac{d\sigma}{dt}-\sigma\right).$$

The equation (3) is equivalent to the sixth Painlevé equation (5.55) [4] with parameters $\alpha=(N-3/2)^2/2$, $\beta=-(N+1/2)^2/2$, $\gamma=1/8$, $\delta=3/8$.

The difference equations for $\langle\sigma_{00}\sigma_{NN}\rangle_{T\pm\lessgtr T_c}$ are written as a first order system. We introduce a set of dependent variables α_N, β_N, etc. tabulated below.

(4) $$G_N^{(0)}=\frac{1}{\alpha_N}\begin{pmatrix}\alpha_{N-1} & \beta_N\\ \gamma_N & \alpha_{N+1}\end{pmatrix},\quad G_N^{(\pm)}=\frac{1}{\alpha_N}\begin{pmatrix}\alpha_N^{(\pm)} & \beta_N^{(\pm)}\\ \gamma_N^{(\pm)} & \delta_N^{(\pm)}\end{pmatrix},$$

where $\det G_N^{(0)}=1$, $\det G_N^{(\pm)}=1$. These quantities (4) satisfy the following bilinear difference equations:

(5) $$\begin{aligned}&\alpha_N\alpha_{N-1}^{(\pm)}-\alpha_{N-1}\alpha_N^{(\pm)}-\beta_{N-1}\gamma_{N-1}^{(\pm)}=0,\ \alpha_N\beta_{N-1}^{(\pm)}-\sqrt{t}^{\mp1}\alpha_{N-1}\beta_N^{(\pm)}-\beta_{N-1}\delta_{N-1}^{(\pm)}=0,\\ &\sqrt{t}^{\mp1}\alpha_N\gamma_{N-1}^{(\pm)}-\alpha_{N-1}\gamma_N^{(\pm)}+\gamma_N\alpha_N^{(\pm)}=0,\ \alpha_N\delta_{N-1}^{(\pm)}-\alpha_{N-1}\delta_N^{(\pm)}+\gamma_N\beta_N^{(\pm)}=0,\\ &(2N+1)\alpha_{N+1}\alpha_{N-1}-(2N-1)\alpha_N^2-\alpha_N^{(+)}\delta_N^{(+)}-\alpha_N^{(-)}\delta_N^{(-)}=0,\\ &(2N-3)\alpha_N\beta_{N-1}-\sqrt{t}^{-1}\alpha_N^{(+)}\beta_N^{(+)}-\sqrt{t}\,\alpha_N^{(-)}\beta_N^{(-)}=0,\\ &(2N-1)\alpha_N\gamma_{N-1}-\gamma_{N-1}^{(+)}\delta_{N-1}^{(+)}-\gamma_{N-1}^{(-)}\delta_{N-1}^{(-)}=0,\\ &(2N+3)\alpha_N\beta_{N+1}-\alpha_{N+1}^{(+)}\beta_{N+1}^{(+)}-\alpha_{N+1}^{(-)}\beta_{N+1}^{(-)}=0,\\ &(2N+1)\alpha_N\gamma_{N+1}-\sqrt{t}^{-1}\gamma_N^{(+)}\delta_N^{(+)}-\sqrt{t}\,\gamma_N^{(-)}\delta_N^{(-)}=0.\end{aligned}$$

The correlation functions are related to (4) through

(6) $$\langle\sigma_{00}\sigma_{NN}\rangle_{T_-<T_c}=t^{-1/4}(t-1)^{1/4}\alpha_{-|N|},\quad \langle\sigma_{00}\sigma_{NN}\rangle_{T_+>T_c}=-t^{-1/4}(t-1)^{1/4}\gamma_{-|N|},$$

where α_N, γ_N correspond to the solution of (5) with the initial condition

(7) $$\begin{aligned}&\alpha_1=\gamma_0^{(-)}=\delta_0^{(+)}=0,\ \alpha_0=\beta_0=-\gamma_0=t^{1/4}(t-1)^{-1/4},\\ &\alpha_0^{(-)}=-\gamma_0^{(+)}=t^{3/4}(t-1)^{-3/4},\ \delta_0^{(-)}=\beta_0^{(+)}=t^{-1/4}(t-1)^{1/4},\\ &\alpha_{-1}=\alpha_0F(-1/2,1/2,1;1/t),\ \beta_0^{(-)}=\alpha_0\sqrt{t-1}(F(-1/2,1/2,1;1/t)\\ &\qquad\qquad -F(1/2,1/2,1;1/t)),\\ &\alpha_0^{(+)}=\alpha_0\sqrt{t}\sqrt{t-1}^{-1}(2F(-1/2,1/2,1;1/t)-F(1/2,1/2,1;1/t)).\end{aligned}$$

2. Spin operators. We use free fermion fields $\psi(\theta)$ and $\psi^\dagger(\theta)$ ($\theta\in\mathbf{R}/2\pi\mathbf{Z}$) satisfying $\langle\psi(\theta)\psi^\dagger(\theta')\rangle=2\pi\delta(\theta-\theta')$. We set $\psi_\pm(\theta)=\psi^\dagger(-\theta)\pm\psi(\theta)$ and define $\varphi_N=:\exp(\rho_N/2):$ by

(8) $$\rho_N/2=\iint\frac{d\theta}{2\pi}\frac{d\theta'}{2\pi}\frac{\sqrt{\omega}}{\sqrt{\omega'}}\frac{e^{iN(\theta+\theta')}}{e^{i(\theta+\theta'-i0)}-1}\psi_+(\theta)\psi_-(\theta')$$

where $\omega=\sqrt{(1-az)(1-az^{-1})}$ ($a=(\sinh\beta_-E_1\sinh\beta_-E_2)^{-1}=1/\sqrt{t}$, $z=e^{i\theta}$). Then, using the results in [8] (Chapter VIII) and [9] we obtain

(9) $$\langle\sigma_{00}\sigma_{NN}\rangle_{T_-<T_c}=(1-a^2)^{1/4}\langle\varphi_0\varphi_N\rangle.$$

The commutator product of the free field $\psi_\pm(\theta)$ and the "spin operator" φ_N is given by

(10) $$[\psi_\pm(\theta),\varphi_N]=2\sqrt{\omega}^{\pm1}z^{1-N}\varphi_N^\mp(z)$$

where $\varphi_N^\pm(z)=:\phi_N^\pm(z)\exp(\rho_N/2):$ and $\phi_N^\pm(z)=\int\frac{d\theta_1}{2\pi}\sqrt{\omega_1}^{\pm1}\psi_\pm(\theta_1)\frac{z_1^N}{z_1-z}$.

We also set $\phi_N^\pm=\phi_{N+1}^\pm(0)=\int\frac{d\theta_1}{2\pi}\sqrt{\omega_1}^{\pm1}\psi_\pm(\theta_1)z_1^N$ and $\varphi_N^\pm=\varphi_{N+1}^\pm(0)=$ $:\phi_N^\pm\exp(\rho_N/2):$. The last identity follows from

(11) $$\varphi_{N+1}-\varphi_N=:\phi_N^+\phi_N^-\exp(\rho_N/2):.$$

The correlation function $\langle\sigma_{00}\sigma_{NN}\rangle_{T_+>T_c}$ is given by

(12) $$\langle\sigma_{00}\sigma_{NN}\rangle_{T_+>T_c}=-(1-\alpha^2)^{1/4}\langle\varphi_0^-\varphi_N^-\rangle$$
with $\alpha=\sinh\beta_+E_1\sinh\beta_+E_2=1/\sqrt{t}$.

3. **Construction of an isomonodromy family.** We define a 2×2 matrix $Y(z,z_0)=\hat{Y}(z,z_0)\begin{pmatrix}\omega z^{-N} & \\ & 1\end{pmatrix}$ by the following series.

(13) $$\hat{Y}(z,z_0)_{\substack{11\\22}}=1+\sum_{l=1}^{\infty}(-\lambda^2)^l\int\cdots\int\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_{2l}}{2\pi}\frac{z-z_0}{z_1-z_0}f_N^{\pm}(z_1,z_2)f_N^{\mp}(z_2,z_3)\cdots f_N^{\mp}(z_{2l},z),$$

(14) $$\hat{Y}(z,z_0)_{\substack{12\\21}}=\pm\lambda\sum_{l=1}^{\infty}(-\lambda^2)^{l-1}\int\cdots\int\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_{2l-1}}{2\pi}\frac{z-z_0}{z_1-z_0}f_N^{\pm}(z_1,z_2)\times f_N^{\mp}(z_2,z_3)\cdots f_N^{\pm}(z_{2l-1},z),$$

where $f_N^{\pm}(z,z')=(\omega/z^N)^{\pm 1}/(1-e^{-i(\theta-\theta'\pm i\kappa 0)})$ with $\kappa=+$ or $-$. $Y(z,z_0)$ is so normalized that $\hat{Y}(z,z_0)=1+O(z-z_0)$ $(z_0\neq\infty)$ or $1+O(1/z)$ $(z_0=\infty)$. Moreover we have $\det\hat{Y}(z,z_0)=1$.

We denote by $\hat{Y}_\pm(z,z_0)$ the restriction of $\hat{Y}(z,z_0)$ to $D_\pm=\{z\,|\,|z|\lessgtr 1\}$, and set $Y_\pm(z,z_0)=\hat{Y}_\pm(z,z_0)\begin{pmatrix}\omega z^{-N} & \\ & 1\end{pmatrix}$. The connection between $Y_\pm(z,z_0)$ is given by

(15) $$Y_-(z,z_0)=Y_+(z,z_0)\begin{pmatrix}1-\lambda^2 & -\lambda\\ \lambda & 1\end{pmatrix}\qquad(\kappa=+),$$
$$Y_-(z,z_0)=Y_+(z,z_0)\begin{pmatrix}1 & -\lambda\\ \lambda & 1-\lambda^2\end{pmatrix}\qquad(\kappa=-).$$

If we modify the expectation value so that $\langle\psi(\theta)\psi^\dagger(\theta')\rangle=\lambda 2\pi\delta(\theta-\theta')$, we obtain the following identities.

(16) $$\hat{Y}_\varepsilon(z,z_0)_{11}=1+\left(1-\frac{z}{z_0}\right)\frac{\langle\varphi_0:\phi_N^-(z)\phi_N^+(z_0^{-1})\exp(\rho_N/2):\rangle}{\langle\varphi_0\varphi_N\rangle},$$
if $N\lessgtr 0$, or $N=0$ and $|z_0|\gtrless 1$, or $N=0$ and $\varepsilon=\pm$ (for $\kappa=\pm$).

(17) $$\hat{Y}_\varepsilon(z,z_0)_{12}=\left(1-\frac{z}{z_0}\right)\frac{\langle\varphi_0^+(z)\varphi_N^+(z_0^{-1})\rangle}{\langle\varphi_0\varphi_N\rangle},$$
if $N\lessgtr 0$, or $N=0$ and $|z_0|\gtrless 1$, or $N=0$ and $\varepsilon=\mp$ (for $\kappa=\pm$).

(18) $$\hat{Y}_\varepsilon(z,z_0)_{21}=\left(1-\frac{z}{z_0}\right)\frac{\langle\varphi_0^-(z_0^{-1})\varphi_N^-(z)\rangle}{\langle\varphi_0\varphi_N\rangle},$$
if $N\lessgtr 0$, or $N=0$ and $|z_0|\lessgtr 1$, or $N=0$ and $\varepsilon=\pm$ (for $\kappa=\pm$).

(19) $$\hat{Y}_\varepsilon(z,z_0)_{22}=1+\left(1-\frac{z}{z_0}\right)\frac{\langle:\phi_0^-(z_0^{-1})\phi_0^+(z)\exp(\rho_0/2):\varphi_N\rangle}{\langle\varphi_0\varphi_N\rangle}$$
if $N\lessgtr 0$, or $N=0$ and $|z_0|\lessgtr 1$, or $N=0$ and $\varepsilon=\mp$ (for $\kappa=\pm$).

(20) $\hat{Y}_+(0,\infty)_{11}=\langle\varphi_0\varphi_{N-1}\rangle/\langle\varphi_0\varphi_N\rangle$ if $\kappa=\pm$, $N\lessgtr 0$,

(21) $\hat{Y}_+(0,\infty)_{12}=\langle\varphi_0^+\varphi_N^+\rangle/\langle\varphi_0\varphi_N\rangle$ if $\kappa=\pm$, $N\lesseqqgtr 0$,

(22) $\hat{Y}_+(0,\infty)_{21}=\langle\varphi_0^-\varphi_N^-\rangle/\langle\varphi_0\varphi_N\rangle$ if $\kappa=\pm$, $N\lesseqqgtr 0$,

(23) $\hat{Y}_+(0,\infty)_{22}=\langle\varphi_0\varphi_{N+1}\rangle/\langle\varphi_0\varphi_N\rangle$ if $\kappa=\pm$, $N\lesseqqgtr 0$.

From (8), (14) and (15) we obtain

(24) $$\frac{d}{da}\log\langle\varphi_0\varphi_N\rangle=-\sum_{\epsilon=\pm}\text{trace}\operatorname*{Res}_{z=a^\epsilon}\hat{Y}_\epsilon(z,a^\epsilon)^{-1}\frac{\partial}{\partial z}\hat{Y}_\epsilon(z,a^\epsilon)\times\frac{\partial}{\partial a}\log\begin{pmatrix}\omega z^{-N} & \\ & 1\end{pmatrix}.$$

4. Deformation and the Schlesinger transformation. The construction in § 3 entails the following monodromy property for the matrix $Y_N(z)=Y_-(z,\infty)$. It is a multi-valued analytic matrix with four regular singularities at $z=0, a, a^{-1}$ and ∞, where the local exponents are given by

(25) $$T_0^{(0)}=\begin{pmatrix}-N-\frac{1}{2} & \\ & 0\end{pmatrix},\ T_0^{(+)}=\begin{pmatrix}\frac{1}{2} & \\ & 0\end{pmatrix},\ T_0^{(-)}=\begin{pmatrix}\frac{1}{2} & \\ & 0\end{pmatrix},\ T_0^{(\infty)}=\begin{pmatrix}N-\frac{1}{2} & \\ & 0\end{pmatrix},$$

respectively. Moreover its global monodromy matrices $\begin{pmatrix}-1 & \\ & 1\end{pmatrix}$ at $z=a^{-1}$, ∞ and $\begin{pmatrix}-1+2\lambda^2 & 2\lambda \\ 2\lambda(1-\lambda^2) & 1-2\lambda^2\end{pmatrix}$ (if $\kappa=+$), $\begin{pmatrix}-1+2\lambda^2 & 2\lambda(1-\lambda^2) \\ 2\lambda & 1-2\lambda^2\end{pmatrix}$ (if $\kappa=-$) at $z=0$, a are independent of $a^{\pm1}$. These properties are sufficient to guarantee that $Y_N(z)$ should satisfy linear differential equations of the form (cf. [3], [5])

(26) $$\frac{\partial Y_N}{\partial z}=\left(\frac{A_0}{z}+\frac{A_+}{z-a}+\frac{A_-}{z-a^{-1}}\right)Y_N$$

$$\frac{\partial Y_N}{\partial a}=\left(-\frac{A_+}{z-a}+\frac{1}{a^2}\frac{A_-}{z-a^{-1}}+\frac{1}{2a}\begin{pmatrix}1 & \\ & 0\end{pmatrix}\right)Y_N$$

(27) $$A_\nu=G_N^{(\nu)}T_0^{(\nu)}G_N^{(\nu)-1}\ (\nu=0,\ \pm),\qquad A_0+A_++A_-=-T_0^{(\infty)}.$$

Here we have set

(28) $$G_N^{(0)}=\hat{Y}_+(0,\infty),\ G_N^{(\pm)}=\hat{Y}_\pm(a^{\pm1},\infty).$$

By a change of variables $z=ax$, $t=a^{-2}$, $Z(x)=KY_N(ax)$, $K=\begin{pmatrix}ia^{N-1} & \\ & 1\end{pmatrix}$, the integrability condition for (26) reduces to a sixth Painlevé equation ((5.55) in [4]) with parameters $\alpha=(N-3/2)^2/2$, $\beta=-(N+1/2)^2/2$, $\gamma=1/8$, $\delta=3/8$. Correlation functions are related to the τ function $\tau_N(t)$ associated with (26); by comparing (24) with the defining equation $d\log\tau_N(t)=\text{trace}\,(A_0A_+(da/a)+A_0A_-(da^{-1}/a^{-1})+A_+A_-(d(a-a^{-1})/(a-a^{-1}))$, we find

(29) $$\langle\varphi_0\varphi_N\rangle=\text{const.}\ t^{1/8}(t-1)^{-1/4}\tau_N(t),$$

The result (3) for $\langle\sigma_{00}\sigma_{NN}\rangle_{T-<T_c}$ follows from (8), (29) and (5.60) [4].

To derive difference equations, we observe that changing N into $N-1$ amounts to shifting the exponents by integers (Schlesinger transformation) as $T_0^{(0)}\mapsto T_0^{(0)}+\begin{pmatrix}1 & \\ & 0\end{pmatrix}$, $T_0^{(\infty)}\mapsto T_0^{(\infty)}+\begin{pmatrix}-1 & \\ & 0\end{pmatrix}$. It is known [4] [5] that such transformations are achieved by multiplication by a rational matrix $R_N(z)$:

(30) $$Y_{N-1}(z)=R_N(z)Y_N(z),\ R_N(z)=\begin{pmatrix}1 & \\ & 0\end{pmatrix}z+R_{0N}$$

$$R_{0N}=\begin{pmatrix}(Y_{1N}^{(\infty)})_{12}(G_N^{(0)})_{21}/(G_N^{(0)})_{11} & -(Y_{1N}^{(\infty)})_{12} \\ -(G_N^{(0)})_{21}/(G_N^{(0)})_{11} & 1\end{pmatrix}=G_{N-1}^{(0)}\begin{pmatrix}0 & \\ & 1\end{pmatrix}G_N^{(0)-1}.$$

Here $Y_{1N}^{(\infty)}$ signifies the coefficient matrix of $\hat{Y}_-(z)=1+Y_{1N}^{(\infty)}z^{-1}+\cdots$ $(z\to\infty)$. In particular, (30) implies

(31) $$G_{N-1}^{(\pm)}=R_N(a^{\pm 1})G_N^{(\pm)}.$$

If we write down (31) and the constraint (27) in terms of the parameters α_N, β_N, $\cdots$ given in (4), we obtain (5). However, care must be taken in identifying α_N, γ_N with $\langle\varphi_0\varphi_N\rangle$ and $\langle\varphi_0^-\varphi_N^-\rangle$. As is shown in (20)–(23), the latter correspond to *different* monodromy problems ($\kappa=+$ or $-$) according to the sign of N, and hence to *different* solutions of (5). This explains the appearance of $|N|$ in (6). Finally the differential equation (3) for $\langle\sigma_{00}\sigma_{NN}\rangle_{T_+>T_c}$ is obtained by noting that $\langle\varphi_0^-\varphi_N^-\rangle$ coincides with the τ function corresponding to the Schlesinger transformation $T_0^{(0)}\to T_0^{(0)}+\begin{pmatrix}1 & \\ & 0\end{pmatrix}$, $T_0^{(\infty)}\to T_0^{(\infty)}+\begin{pmatrix}0 & \\ & -1\end{pmatrix}$ (Theorem 4.1 [4]).

5. Scaling limit. Here we shall show that the previously known results [1] are reproduced from (3), (5) in the scaling limit $N\to\infty$, $t=1+N^{-1}\bar{t}$ with $\bar{t}>0$ fixed.

In this limit the confluence of two regular singularities $z=0$, ∞ takes place to produce an irregular singularity of rank 1.

Since the monodromy stays constant as we vary N, the limiting monodromy data are determined from the original ones. To see this we scale the infinite series (14) by setting $z=e^{i\varepsilon p}$, $\varepsilon=\bar{t}/2mN$ $(m>0$: arbitrary). Choosing $\kappa=+$ we then have

(32) $$\lim_{\varepsilon\to 0}\begin{pmatrix}\varepsilon^{-1} & \\ & 1\end{pmatrix}Y_{\pm,-N}(z,\infty)=\bar{Y}_\pm(p)=\hat{\bar{Y}}_\pm(p)\begin{pmatrix}\bar{\omega}(p)e^{i\bar{t}p\kappa m} & \\ & 1\end{pmatrix},$$

where $\bar{\omega}(p)=\sqrt{p^2+m^2}$ and $\hat{\bar{Y}}_\pm(p)=1+O(p^{-1})$ as $p\to\infty$ in the region $\mathscr{D}_\pm$ (Fig. 1).

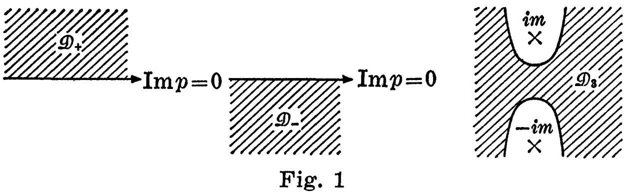

Fig. 1

Modifying (14) slightly we get also, after scaling, $\bar{Y}_3(p)$ which has a similar property in the region $\mathscr{D}_3$. These are connected through

(33) $$\bar{Y}_+(p)=\bar{Y}_3(p)\begin{pmatrix}1 & \lambda \\ 0 & 1\end{pmatrix},\ \bar{Y}_-(p)=\bar{Y}_3(p)\begin{pmatrix}1 & 0 \\ \lambda & 1\end{pmatrix}.$$

The linear differential equations (26) tend to

(34) $$\frac{\partial\bar{Y}}{\partial p}=\left(\frac{\bar{A}_+}{p-im}+\frac{\bar{A}_-}{p+im}+\begin{pmatrix}i\bar{t}/2m & \\ & 1\end{pmatrix}\right)\bar{Y},\ \frac{\partial\bar{Y}}{\partial\bar{t}}=\frac{1}{2m}\begin{pmatrix}ip & \bar{b} \\ -\bar{c} & 0\end{pmatrix}\bar{Y}$$

(35) $$\bar{A}_{\pm}=\bar{G}^{(\pm)}T_0^{(\pm)}\bar{G}^{(\pm)-1},\ \bar{A}_{+}+\bar{A}_{-}=\begin{pmatrix}1 & \bar{t}\bar{b}/2m\\ -\bar{t}\bar{c}/2m & 0\end{pmatrix}$$

$$\bar{G}^{(\pm)}=\begin{pmatrix}\bar{a}^{(\pm)} & \bar{b}^{(\pm)}\\ \bar{c}^{(\pm)} & \bar{d}^{(\pm)}\end{pmatrix}=\lim_{\varepsilon\to 0}\begin{pmatrix}\varepsilon & \\ & 1\end{pmatrix}G^{(\pm)}\begin{pmatrix}\varepsilon^{-1} & \\ & 1\end{pmatrix}.$$

Here we have set $\bar{b}=\lim \varepsilon^{-2}\beta_N/\alpha_N$, $\bar{c}=\lim \gamma_N/\alpha_N$.

The difference equations (5) are scaled to give

(36) $$\frac{d}{d\bar{t}}\left(\bar{G}^{(\pm)}\begin{pmatrix}e^{\mp\bar{t}/2} & \\ & 1\end{pmatrix}\right)=\begin{pmatrix}\mp 1/2 & \bar{b}/2m\\ -\bar{c}/2m & 0\end{pmatrix}\bar{G}^{(\pm)}\begin{pmatrix}e^{\mp\bar{t}/2} & \\ & 1\end{pmatrix},$$

which is one of the equivalent expressions of the deformation equations for (34).

If we set $\bar{A}_{+}=\begin{pmatrix}1/2+\bar{z} & -\bar{u}(1/2+\bar{z})\\ \bar{z}/\bar{u} & -\bar{z}\end{pmatrix}$, $\bar{A}_{-}=\begin{pmatrix}1/2-\bar{z} & \bar{u}\bar{y}(-1/2+\bar{z})\\ -\bar{z}/\bar{u}\bar{y} & \bar{z}\end{pmatrix}$, then $\bar{y}=\bar{y}(\bar{t})$ is a solution of PV with $\alpha=1/8$, $\beta=-1/8$, $\gamma=0$, $\delta=-1/2$. The relation (29) reduces in the limit to (4.11.9) [1] (with $\bar{t}=-t$; the factor 1/2 there is erroneous)

(37) $$\lim d\log\langle\varphi_0\varphi_N\rangle=\left(-\bar{t}\bar{z}+\left(-2\bar{z}^2-\bar{y}\bar{z}\left(\frac{1}{2}-\bar{z}\right)+\bar{y}^{-1}\bar{z}\left(\frac{1}{2}+\bar{z}\right)\right)\right)\frac{d\bar{t}}{\bar{t}}.$$

Finally the differential equation (3) is scaled to

(38) $$\left(\bar{t}\frac{d^2\bar{\sigma}}{d\bar{t}^2}\right)^2=\left(\bar{\sigma}-\bar{t}\frac{d\bar{\sigma}}{d\bar{t}}+2\left(\frac{d\bar{\sigma}}{d\bar{t}}\right)^2\right)^2+4\left(\frac{d\bar{\sigma}}{d\bar{t}}\right)^2\left(\frac{1}{4}-\left(\frac{d\bar{\sigma}}{d\bar{t}}\right)^2\right)$$

where $\bar{\sigma}(\bar{t})=\lim \sigma_{N,\pm}(t)$.

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