94. On Determinants of Cartan Matrices of p-Blocks

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1. Introduction. Let B be a p-block of a finite group with defect group D, and C_B the Cartan matrix of B. Then it is known that $\det C_B \geq |D|$. In [6] we showed that the equality holds in the above under some assumption. The purpose of this note is to extend this result.

Notation. Let G be a finite group with order divisible by a fixed prime p and $\mathfrak{p}$ a fixed prime divisor of p in the ring $\boldsymbol{Z}[\varepsilon]$, where ε is a primitive $|G|$-th root of 1. We denote by F the residue class field $\boldsymbol{Z}[\varepsilon]/\mathfrak{p}$, by FG the group algebra of G over F, and by $Z(FG)$ the center of FG. If B is a block of G, we denote by C_B the Cartan matrix of B, by $D(B)$ a defect group of B, and by $l(B)$ the number of irreducible modular characters in B. If Q is a p-subgroup of G, $m_B(Q)$ denotes the number of p-regular (conjugate) classes of G associated with B which have Q as a defect group. (For selection of sets of conjugate classes for the blocks, see Brauer [1], [2], [4], Osima [8], and Iizuka [7].) We denote by $S(B)$ the set of subsections $s=(\pi, b)$ associated with B which are different from $1=(1, B)$. (For a subsection, see Brauer [3].) For brevity we write $C(X)$ and $N(X)$ instead of $C_G(X)$ and $N_G(X)$ for a subset X of G respectively. If K is a conjugate class of G, we denote by $\hat{K}$ the class sum of K in the group algebra FG.

The main result of this note is the following

Theorem. *Let B be a block of G.*

(i) *For a proper subgroup $Q \neq 1$ of $D(B)$, if $m_B(Q) \neq 0$, then $m_b(Q) \neq 0$ for some $s=(\pi, b) \in S(B)$ such that $D(b)$ contains Q as a proper subgroup.*

(ii) *If $\det C_b = |D(b)|$ for any $s=(\pi, b) \in S(B)$, then $\det C_B = |D(B)|$.*

Next corollary is an immediate consequence of Theorem, (ii).

Corollary 1. *Let B be a block of G with defect group D. Suppose that $l(b)=1$ for any $s=(\pi, b) \in S(B)$. Then $\det C_B = |D|$.*

As a special case we have the following

Corollary 2 (Fujii [6]). *Let B be a block of G with defect group D. Suppose that the centralizer in G of any element of order p of D is p-nilpotent. Then $\det C_B = |D|$.*

Remark. The first part of Theorem still holds even if we denote by $m_B(Q)$ the number of conjugate classes of G associated with B which have Q as a defect group.

Remark. For any $s=(\pi, b)\in S(B)$, $\det C_b=|D(b)|$ is equivalent to $l(b)=1$.

2. Proof of Theorem. Let $D(B)=D$. The defect groups of p-regular classes of G associated with B are all conjugate to subgroups of D, and the set of their orders coincides with the set of elementary divisors of C_B. The greatest elementary divisor of C_B is equal to $|D|$ and all other elementary divisors are less than $|D|$. (For example, see Curtis-Reiner [5] and Brauer [4].) Therefore (i) implies (ii). We shall prove (i).

In the proof of (i), the next lemma is fundamental.

Lemma (Brauer [4]). *Let B be a block of G with defect group D. For any subgroup Q of D, $m_B(Q)=\Sigma_{\tilde{B}} m_{\tilde{B}}(Q)$, where $\tilde{B}$ ranges over the blocks of $N(Q)$ with $\tilde{B}^G=B$.*

By the lemma, there exists a block $\tilde{B}$ of $N(Q)$ with $\tilde{B}^G=B$ such that $m_{\tilde{B}}(Q)\neq 0$. By Brauer's first main theorem we may assume that $Q\subsetneqq D(\tilde{B})\subset D(B)$. Let $E=\Sigma_K a_K\hat{K}$ be the block idempotent of $Z(FN(Q))$ corresponding to $\tilde{B}$, where K ranges over the p-regular classes of $N(Q)$ and $a_K\in F$. If $a_K\neq 0$ then a defect group of K contains Q. Therefore E is an idempotent of $Z(FQC(Q))$. Let $E=e_0+\cdots$ be the decomposition into the sum of block idempotents of $Z(FQC(Q))$ and b_0 the block of $QC(Q)$ corresponding to e_0. Then $b_0^{N(Q)}=\tilde{B}$ and $\tilde{B}$ is a unique block which covers b_0. Let T denote the inertia group of b_0 in $N(Q)$. Then e_0 is also a block idempotent of $Z(FT)$ corresponding to the unique block b_0^T which has $D(\tilde{B})$ as a defect group and covers b_0. Therefore if H is a subgroup of $N(Q)$ which contains $D(\tilde{B})C(Q)$, it follows that b_0^H is a unique block which has $D(\tilde{B})$ as a defect group and covers b_0.

Now we may choose a p-element $\pi(\neq 1)$ such that $\pi\in Q\cap Z(S)$, where S is a Sylow p-subgroup of T which contains $D(\tilde{B})$, since Q is normal in S and so $Q\cap Z(S)\neq 1$. Let $H=C(\pi)\cap N(Q)$. Since $S\subset H$, we have that $(p, |T:H\cap T|)=1$. Since $E\in Z(FQC(Q))$, $E\in Z(FH)$ and let $E=E_0+\cdots$ be the decomposition into the sum of block idempotents of $Z(FH)$ and B_0 the block of H corresponding to E_0. Then we may assume that B_0 covers b_0, so $B_0=b_0^H$ and $D(B_0)=D(\tilde{B})$.

Since $m_{\tilde{B}}(Q)\neq 0$, there exists a p-regular class K of $N(Q)$ with defect group Q such that $\hat{K}E\neq 0$. Since $K\subset QC(Q)$, K is a union of some p-regular classes $\{L\}$ of H with defect group Q. Then we have $\hat{K}E_0\neq 0$ and this means $\hat{L}E_0\neq 0$ for some L. Indeed $\hat{K}E\neq 0$ implies $\hat{K}e_0\neq 0$. Let $\{x_i\}$ be an $(H\cap T)$-transversal of H and $\{y_j\}$ an H-transversal of $N(Q)$. Now assume that $\hat{K}E_0=0$. Then it follows that $0=\Sigma_j\hat{K}E_0^{y_j}=\Sigma_{i,j}\hat{K}e_0^{x_iy_j}=|T:H\cap T|\hat{K}E$. Since $|T:H\cap T|\not\equiv 0 \pmod{p}$, this means $\hat{K}E=0$, a contradiction. Therefore this implies $m_{B_0}(Q)\neq 0$.

Now let $b=B_0^{G(\pi)}$ and $s=(\pi, b)$. Since $b^G=B_0^G=B$, we have $s=(\pi, b) \in S(B)$. Then we have $m_b(Q)\neq 0$ by the lemma since $m_{B_0}(Q)\neq 0$.

References

[1] R. Brauer: On the arithmetic in a group ring. Proc. Nat. Acad. Sci., **30**, 109–114 (1944).

[2] ——: On blocks of characters of groups of finite order I. Ibid., **32**, 182–186 (1946).

[3] ——: On blocks and sections in finite groups I, II. Amer. J. Math., **89**, 1115–1136 (1967); **90**, 895–925 (1968).

[4] ——: Defect groups in the theory of representations of finite groups. Illinois J. Math., **13**, 53–73 (1969).

[5] C. W. Curtis and I. Reiner: Representation theory of finite groups and associative algebras. Wiley-Interscience, New York (1962).

[6] M. Fujii: A remark on the Cartan matrix of a certain p-block. Osaka J. Math., **17**, 411–414 (1980).

[7] K. Iizuka: Notes on blocks of group characters. Kumamoto J. Sci., Ser. A2, 309–321 (1956).

[8] M. Osima: Notes on blocks of group characters. Math. J. Okayama Univ., **4**, 175–188 (1955).