# 91. *Free Arrangements of Hyperplanes and Unitary Reflection Groups**)

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**1. Free arrangements.** We call a non-void finite family of hyperplanes in $C^{n+1}$ (or $P^{n+1}(C)$) an *affine* (resp. *projective*) *n-arrangement*. A set $X$ is simply called an *n-arrangement* if $X$ is either an affine $n$-arrangement or a projective $n$-arrangement. An $n$-arrangement $X$ is called to be *central* when $\bigcap_{H\in X} H \neq \phi$. Denote $\bigcup_{H\in X} H$ by $|X|$.

Let $X$ be a central affine $n$-arrangement. By an appropriate translation of the origin we can assume that $\bigcap_{H\in X} H$ contains the origin $O$ in $C^{n+1}$. Let $Q\in C[z_0,\cdots,z_n]$ be a square-free defining equation of $|X|$. By $\mathcal{O}$ denote we $\mathcal{O}_{C^{n+1},O}$. Then

$$D(X):=\{\theta\,;\ \text{a germ at the origin of holomorphic vector fields such that } \theta\cdot Q\in Q\cdot\mathcal{O}\}$$

is an $\mathcal{O}$-module. We call $X$ to be *free* if $D(X)$ is a free $\mathcal{O}$-module.

Assume that a central affine $n$-arrangement $X$ is free. Let $\{\theta_0,\cdots,\theta_n\}$ be a system of free basis for $D(X)$ such that each $\theta_i$ is homogeneous of degree $d_i$. ($\theta_i$ is homogeneous of degree $d_i$ if $\theta_i$ has an expression

$$\theta_i=\sum_{j=0}^{n} f_j(\partial/\partial z_j),$$

where each $f_j\in C[z_0,\cdots,z_n]$ is either 0 or homogeneous of degree $d_i$.) We call the integers $(d_0,\cdots,d_n)$ the *generalized exponents* of $X$. They depend only on $X$ [7].

Let $X$ be a projective $n$-arrangement. Denote $P^{n+1}(C)$ simply by $P^{n+1}$. Let $Q\in C[z_0,\cdots,z_{n+1}]$ be a homogeneous polynomial defining a set $|X|\subset P^{n+1}$. Then there exists a unique central affine $(n+1)$-arrangement $\tilde{X}$ such that

$$V(Q)=|\tilde{X}|\subset C^{n+2}.$$

We call $X$ to be *free* if $\tilde{X}$ is free.

Assume that a projective $n$-arrangement $X$ is free. Let $(d_0, d_1, \cdots, d_n)$ be the generalized exponents of $\tilde{X}$, then we can assume that $d_0=1$ (due to the existence of the Euler vector field

$$\sum_{i=0}^{n} z_i(\partial/\partial z_i)).$$

The *generalized exponents* of $X$ are defined to be $(d_1,\cdots,d_n)$.

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Next let $X$ be a (perhaps non-central) affine $n$-arrangement. Identify $\boldsymbol{C}^{n+1}$ with a Zariski open set $\boldsymbol{P}^{n+1}\backslash H_\infty$, where $H_\infty$ is a hyperplane in $\boldsymbol{P}^{n+1}$. Define a projective $n$-arrangement

$$X_\infty = X \cup \{H_\infty\}.$$

We call $X$ to be *free* if $X_\infty$ is free. Assume that a projective $n$-arrangement $X$ is free. Then the *generalized exponents* of $X$ are defined to be those of $X_\infty$. This definition is consistent with that of the generalized exponents of a free central affine $n$-arrangement.

We have thus defined the generalized exponents of any free $n$-arrangement. Let $X$ be an $n$-arrangement and $(d_0, \cdots, d_n)$ be its generalized exponents. Put

$$M = \begin{cases} \boldsymbol{C}^{n+1}\backslash|X| & \text{(when } X \text{ is affine)} \\ \boldsymbol{P}^{n+1}\backslash|X| & \text{(when } X \text{ is projective).} \end{cases}$$

Let $P_M(t)$ be the Poincaré polynomial of $M$. Then we have

**Theorem 1.** $P_M(t) = \prod_{i=0}^{n} (1+d_i t)$.

The proof of Theorem 1 highly depends upon the combinatorial formula (using the Möbius functions) for $P_M(t)$ ([2, (5.2)], [9, Theorem A]) and the theory of the Hilbert polynomial of $\mathcal{O}/J(Q)$. ($J(Q)$ stands for the Jacobian ideal of $Q$ in $\mathcal{O}$.) The complete proof will be found in [8] [9].

Let $G \subset \mathrm{GL}(n+1; \boldsymbol{R})$ be a finite Coxeter group acting on $\boldsymbol{C}^{n+1}$. Then the set of the reflection hyperplanes makes a central affine arrangement. Such an arrangement is called a *Coxeter arrangement*. We know that a Coxeter arrangement is free and that the exponents of $G$ coincide with our generalized exponents of $X$ [4]. In this special case Theorem 1 was obtained by Shepherd-Todd-Brieskorn [6] [1].

But the class of the free central affine arrangement is far wider than that of the Coxeter arrangements. In fact many examples show that the freeness of arrangement is a combinatorial property [7].

The following theorem gives another important class of free central arrangements:

**Theorem 2.** *Let $G \subset \mathrm{GL}(n+1; \boldsymbol{C})$ be a finite group generated by unitary reflections. Then the set of the reflection hyperplanes makes a free central affine arrangement.*

This will be proved in the following section.

**2. The proof of Theorem 2.** Put $V = \boldsymbol{C}^{n+1}$. We regard $V$ as a unitary space with the ordinary hermitian form $\sum_{i=0}^{n} x_i \bar{y}_i$. Let $G \subset \mathrm{U}(n+1)$ be a finite group generated by unitary reflections. Put

$$e_i = {}^t[0, \cdots, 0, \underset{\substack{\uparrow \\ i\text{-th place}}}{1}, 0, \cdots, 0] \qquad (0 \le i \le n),$$

and $\{e_0, \cdots, e_n\}$ is a system of orthonormal basis for $V$. Then $g \in G$ acts on $V$ by

$$[g\cdot e_0, \cdots, g\cdot e_n]=[e_0, \cdots, e_n]\cdot g,$$

i.e., $g\cdot e_j=\sum_{i=0}^{n} g_{ij}e_i$ ($g_{ij}$ is the $(i, j)$-entry of $g$).

Let $V^*$ be the dual $C$-vector space of $V$. Let $\{z_0, \cdots, z_n\}\subset V^*$ be the system of the dual basis of $\{e_0, \cdots, e_n\}$. Then $g\in G$ acts on $V^*$ by

$$[g\cdot z_0, \cdots, g\cdot z_n]=[z_0, \cdots, z_n]\cdot {}^t g^{-1},$$

which is the contragradient representation. This representation of $G$ induces another representation of $G$ on $S=S(V^*)$ (the symmetric product of $V^*$) by

$$g\cdot f(z_0, \cdots, z_n)=f(g\cdot z_0, \cdots, g\cdot z_n) \qquad (f\in S\simeq C[z_0, \cdots, z_n]).$$

Thus $g\in G$ acts on $S\otimes V$ by

$$f\otimes v\mapsto (g\cdot f)\otimes(g\cdot v) \qquad (f\in S,\ v\in V).$$

In this situation, there exist $u_0, \cdots, u_n\in (S\otimes V)^G$ such that

$$(S\otimes V)^G=S^G u_0\oplus\cdots\oplus S^G u_n$$

and each $a_{ij}\in S$ $(0\leq j\leq n)$ is a homogeneous polynomial of $z_0, \cdots, z_n$ of degree $d_i$, where

$$u_i=\sum_{j=0}^{n} a_{ij}\otimes e_j \qquad (0\leq i\leq n).$$

Define $\Delta\in S$ by

$$\Delta=\det(a_{ij}).$$

Let $X$ be the set of the reflection hyperplanes of $G$. The following proposition was proved by Orlik-Solomon [3]:

**Proposition 1.** (i) *$\Delta$ is a square-free defining equation of $|X|$.*

(ii) *Let $f\in S$. Then $g\cdot f=(\det g)^{-1}\cdot f$ for any $g\in G$ if and only if $f\in S^G\cdot\Delta$.*

Define vector fields on $V$ by

$$\mathfrak{X}_i=\sum_{j=0}^{n} a_{ij}(\partial/\partial z_j) \qquad (0\leq i\leq n).$$

**Proposition 2.** *A set $\{\mathfrak{X}_0, \cdots, \mathfrak{X}_n\}$ is a system of free basis for $D(X)$.*

**Proof.** Since $u_i$ is invariant under $G$, we have

$$\sum_j a_{ij}\otimes e_j=\sum_j (g\cdot a_{ij})\otimes(g\cdot e_j) \qquad (0\leq i\leq n)$$

and thus

$$(*) \qquad (a_{ij})=(g\cdot a_{ij})\cdot {}^t g.$$

Then

$$\begin{aligned}
&[g\cdot(\mathfrak{X}_0\cdot\Delta), \cdots, g\cdot(\mathfrak{X}_n\cdot\Delta)]\\
&\quad=[\partial(g\cdot\Delta)/\partial(g\cdot z_0), \cdots, \partial(g\cdot\Delta)/\partial(g\cdot z_n)]\cdot {}^t(g\cdot a_{ij})\\
&\quad=(\det g)^{-1}[\partial\Delta/\partial(g\cdot z_0), \cdots, \partial\Delta/\partial(g\cdot z_n)]\cdot g^{-1}\cdot {}^t(a_{ij})\\
&\qquad\qquad\qquad\text{(by Proposition 1 (ii) and } (*))\\
&\quad=(\det g)^{-1}[\partial\Delta/\partial z_0, \cdots, \partial\Delta/\partial z_n]\cdot(\partial z_i/\partial(g\cdot z_j))\cdot g^{-1}\cdot {}^t(a_{ij})\\
&\quad=(\det g)^{-1}[\partial\Delta/\partial z_0, \cdots, \partial\Delta/\partial z_n]\cdot {}^t(a_{ij})\\
&\qquad\qquad\qquad\text{(by } g=(\partial z_i/\partial(g\cdot z_j)))\\
&\quad=(\det g)^{-1}[\mathfrak{X}_0\cdot\Delta, \cdots, \mathfrak{X}_n\cdot\Delta].
\end{aligned}$$

By combining this with Proposition 1 (ii), we have

$$\mathfrak{X}_i \cdot \Delta \in S^G \cdot \Delta \qquad (0 \leq i \leq n).$$

This implies that $\mathfrak{X}_i \in D(X)$ $(0 \leq i \leq n)$ because of Proposition 1 (i). Since $\Delta = \det(a_{ij})$ is a square-free defining equation of $X$, we know that a set $\{\mathfrak{X}_0, \cdots, \mathfrak{X}_n\}$ is a system of free basis for $D(X)$ in the light of [5, (1.8) ii)].

The following is obtained from Theorems 1 and 2:

**Corollary.** *Put* $d_i = \deg \mathfrak{X}_i$ $(0 \leq i \leq n)$. *Then the Poincaré polynomial of* $C^{n+1} \backslash |X|$ *is equal to*

$$\prod_{i=0}^{n} (1 + d_i t).$$

This result was very recently proved by Orlik-Solomon [3]. Thus Theorem 1 was proved to be a generalization of the main theorem in [3].

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