90. *On the Linear Sieve. II*

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1. The purpose of the present note is to show briefly an alternative proof of Iwaniec's remarkable improvement [2] upon the linear sieve of Rosser. Our argument is not much different from Iwaniec's, but, being a straightforward refinement of [3], it is comparatively more direct and easy. Roughly speaking, our procedure is an injection of a smoothing device to Rosser's infinite iteration of the Buchstab identity.

We retain most of the notations of [3], and in addition we introduce the condition Ω_∞: For any $3\le u<v$

$$\sum_{u\le p<v}\delta(p^2)p^{-2}=O((\log\log u)^{-1}).$$

Then the linear sieve of Iwaniec is, in a modified form,

Theorem. *Provided* $MN\ge z^2$, Ω_∞, $\Omega_2(1,L)$, $L\le(\log z)/(\log\log z)$, *we have, for* $\nu=0$ *and* 1,

$$(-1)^{\nu-1}\left\{S(A,z)-\left(\phi_\nu\left(\frac{\log MN}{\log z}\right)+O((\log\log z)^{-1/50})\right)XV(z)\right\}$$
$$\le\log z\,\operatorname*{Max}_{\alpha,\beta}\left|\sum_{\substack{m<M\\ n<N}}\alpha_m\beta_nR_{mn}\right|,$$

where $\{\alpha_m\}$, $\{\beta_n\}$ *are variable vectors such that* $|\alpha_m|\le1$, $|\beta_n|\le1$.
Detailed discussions will be given in [5], and here we indicate only the clues. Note that we have obtained a hybrid of this result (but $\nu=1$ only) with the multiplicative large sieve (see [4]).

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2. To state our principal lemmas we introduce the following conventions: We put $z=z_1z_2^J$, where J is a large integer to be specified later. We dissect $[z_1,z)$ into J smaller intervals $[z_1z_2^{j-1},z_1z_2^j)$, and denote one of them generally by I with or without suffix. K with or without suffix stands for the set-theoretic direct product of a sequence of I's, and $\omega(K)$ be the number of constituent I's. If $K=I_1I_2\cdots I_r$ then $I<K$ means that $(I)<\min(I_j)$ $(j\le r)$, where (I) is the right end point of I; also $d\in K$ implies that $d=p_1p_2\cdots p_r$ with $p_j\in I_j$, $p_j\in P$. Note that we do not reject non-squarefree d. Next we define Φ_ν and Γ_ν $(\nu=0,1)$ to be the characteristic functions of the sets of K such that

$$\{K=I_1I_2\cdots I_r \mid I_1>I_2>\cdots>I_r,\quad (I_1)(I_2)\cdots(I_{2k+\nu-1})(I_{2k+\nu})^3<y \quad (1\le 2k+\nu\le r)\}$$

and

$$\{K=I_1I_2\cdots I_r \mid I_1>I_2>\cdots>I_r,\quad r\equiv\nu \pmod 2,\quad \Phi_\nu(I_1I_2\cdots I_{r-1})=1,\quad (I_1)(I_2)\cdots(I_{r-1})(I_r)^3\ge y\},$$

respectively. Then generalizing the Rosser truncation of the iteration of the Buchstab identity we get easily

Lemma 1.

$$\begin{aligned}S(A,z)=&\sum_K(-1)^{\omega(K)}\Phi_\nu(K)\sum_{d\in K}S(A_d,z_1)\\&+\sum_{\substack{I,K\\I<K}}(-1)^{\omega(K)}\Phi_\nu(KI)\sum_{\substack{p'<p\\p',p\in I\\d\in K}}S(A_{dpp'},p')\\&+(-1)^\nu\sum_K\Gamma_\nu(K)\sum_{d\in K}S(A_d,p(d)).\end{aligned}$$

Corollary.

$$\begin{aligned}(-1)^\nu S(A,z)\ge&(-1)^\nu\sum_K(-1)^{\omega(K)}\Phi_\nu(K)\sum_{d\in K}S(A_d,z_1)\\&-\sum_{\substack{I,K\\I<K\\\omega(K)\equiv\nu+1\,(\mathrm{mod}\,2)}}\Phi_\nu(KI)\sum_{\substack{p,p'\in I\\d\in K}}S(A_{dpp'},z_1).\end{aligned}$$

Remark. Note that the condition $p'<p$ has been dropped out. With a little effort we can prove the following modification of Lemma 1 of [3]:

Lemma 2. *Provided* $\Omega_2(1,L)$, $z\le y^{1/2(1+\nu)}$, *we have, for* $\nu=0,1$,

$$\begin{aligned}V(z)\phi_\nu\left(\frac{\log y}{\log z}\right)=&V(z_1)\sum_K(-1)^{\omega(K)}\Phi_\nu(K)\sum_{d\in K}\frac{\delta(d)}{d}\phi_{\nu+\omega(K)}\left(\frac{\log y/d}{\log z_1}\right)\\&+O\left\{V(z)\frac{(\log z)^2}{(\log z_1)^3}\left(L+(\log z_2)\log\left(\frac{\log z}{\log z_1}\right)\right)\right\}.\end{aligned}$$

Next we state in a slightly more precise form the crucial observation of Iwaniec [2, Lemma 3]:

Lemma 3. *Let* $y=MN\ge z^2$. *Then* $\Phi_\nu(K)=1$ *implies that there exists a decomposition* $K=K_1K_2$ *such that* $(K_1)<M$, $(K_2)<N$. *Also if* $\Phi_\nu(KI)=1$, $I<K$ *and* $\omega(K)\equiv\nu+1 \pmod 2$, *then, since* $\Phi_\nu(K)=1$, *we have a decomposition* $K=K_1K_2$ *as above, and at least one of the following three cases occurs:* $\{(K_1)(I)<M,\ (K_2)(I)<N\}$, $\{(K_1)(I)^2<M,\ (K_2)<N\}$, $\{(K_1)<M,\ (K_2)(I)^2<N\}$. *Here e.g.* (K_1) *is the product of* (I)*'s of* I*'s appearing in* K_1.

3. Now we indicate the main steps of our proof of the theorem. Set $z_1=z^{\varepsilon^2}$, $z_2=z^{\varepsilon^9}$ with $\varepsilon=(\log\log z)^{-1/10}$; thus $J\le(\log\log z)^{9/10}$. Also we assume $L\le(\log z)/(\log\log z)$. By the fundamental lemma (see [1, Lemma 5] for its quick proof) we see that there are two sequences $\{h_\tau^{(\nu)}\}$ $(\nu=0,1)$ such that $h_\tau^{(\nu)}=0$ for $\tau\nmid P(z_1)$, and $|h_\tau^{(\nu)}|\le 1$, and such that uniformly for $d\nmid P(z_1)$

$$(-1)^\nu\left\{S(A_d,z_1)-\frac{\delta(d)}{d}XV(z_1)(1+O(e^{-H}))\right\}$$

$$\geq (-1)^{\nu} \sum_{\tau < z_1^H}{}' h_\tau^{(\nu)} R_{d\tau},$$

where H is at our disposal. We set $H=\varepsilon^{-1}$, and apply these inequalities to the right side of the corollary to Lemma 1 above. We estimate the resulting main term by appealing to the condition Ω_∞ and to Lemma 2 above as well as Lemma 2 of [3] (but with more accurate $O(L(\log z)(\log z_1)^{-2})$ in the place of $O(L(\log y)^{-2/5})$). Then the argument of [3] can be carried into our new situation without alternation, except for the choice of B appearing there; here we set $3^B=\varepsilon^{-1}$. In this way we get the main term of the theorem. As for the error term we see readily that its absolute value is less than the expression

$$\sum_K{}' \Phi_\nu(K) \Bigl| \sum_{\substack{d\in K \\ \tau < z^\varepsilon}}{}' h_\tau^{(\nu+\omega(K))} R_{d\tau} \Bigr| + \sum_{\substack{I,K \\ I<K \\ \omega(K)\equiv \nu+1 \pmod 2}} \Phi_\nu(KI) \Bigl| \sum_{\substack{d\in K \\ p',p\in I \\ \tau<z^\varepsilon}} h_\tau^{(1)} R_{d\tau pp'} \Bigr|,$$

in which $h_\tau^{(j)}=h_\tau^{(\nu)}$ if $j\equiv\nu \pmod 2$. Then Lemma 3 gives the desired decompositions of d or dpp' in these sums. Finally noticing that the number of permissible K is less than $2^J<\log z$ we conclude the proof of the theorem.

References

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