88. *Calculus on Gaussian White Noise. I*

By Izumi KUBO and Shigeo TAKENAKA
Department of Mathematics, Faculty of Sciences,
Nagoya University

(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 13, 1980)

§ 1. **Introduction.** Recently, Hida has introduced *generalized Brownian functionals* to discuss the analysis on the L^2-space (L^2) built on the measure space of white noise $\dot{B}(t)$. The idea of Hida's analysis is to take $\{\dot{B}(t)\}$ to be the system of the variables of Brownian functionals, so that we are led to introduce multiplication operators $\dot{B}(t)$ and the partial differential operators $\partial/\partial\dot{B}(t)$ as well as renormalization of functions of the $\dot{B}(t)$'s [1, 2]. We will give, in this series of notes Parts I–V, a systematic treatment of his analysis and establish formulae which would make easier to apply his theory.

We will discuss, in Part I, a general theory on Fock spaces and Hilbert spaces of non-linear functionals of special types, which is a slight modification of the works of Segal [3], [4] and of Hida-Ikeda [5].

In Part II, the L^2-space $(L^2)=L^2(\mathcal{E}^*, \mu)$ will be discussed, where $\mathcal{E}\subset E_0\subset\mathcal{E}^*$ is a Gelfand triplet and μ is the measure of Gaussian white noise on $\mathcal{E}^*$. With the help of transformation $\mathcal{S}$,

$$(\mathcal{S}\varphi)(\xi)=\int_{\mathcal{E}^*}\varphi(x+\xi)d\mu(x),\ \xi\in\mathcal{E},\ \varphi\in(L^2),$$

we can apply the analysis established in Part I. We will treat operators $\partial/\partial x(t)$, $(\partial/\partial x(t))^*$, $x(t)\cdot=\partial/\partial x(t)+(\partial/\partial x(t))^*$ and so forth to carry on the proposed analysis of Brownian functionals.

In Part III, we will describe Hida's analysis by our formulation, partly. In Part IV, Laplacians on (L^2) will be discussed. In Part V, we will discuss Hida-Streit's approach to Feynman path integral in line with our formulation.

§ 2. **Triplets of Fock spaces.** Let $(E_0, (\xi, \eta)_0)$ be a separable real Hilbert space and let us identify its dual E_0^* with E_0. Suppose that $\mathcal{E}$ is a dense linear subset of E_0. Let $\{(\ ,\ )_p\,;\, p\geq 0\}$ be a consistent sequence of inner products defined on $\mathcal{E}$ such that

(2.1) $\|\xi\|_0\leq\rho\|\xi\|_1\leq\cdots\leq\rho^p\|\xi\|_p\ \cdots,$ with $\rho\in(0,1)$.

Let E_p be the completion of $\mathcal{E}$ in $\|\ \ \|_p$, and $E_{-p}\equiv E_p^*$ be the dual of E_p with the inner product $(\ ,\ )_{-p}$, for $p>0$. Then we have inclusions

$$\cdots\subset E_{p+1}\subset E_p\subset\cdots\subset E_0\subset\cdots\subset E_{-p}\subset E_{-p-1}\cdots.$$

Let E_∞ be the projective limit of the system $\{(E_p, \|\ \ \|_p)\,;\, p\in Z\}$. Suppose that $\mathcal{E}=E_\infty$ as a set and induce the topology by this equality.

The dual $\mathcal{E}^*$ of $\mathcal{E}$ is the inductive limit $E_{-\infty}$ of the system $\{(E_p, \|\ \|_p)\}$. Denote the natural injection from E_q to E_p by $\iota_{p,q}$, $q>p$. Then an isomorphism θ_p from E_p^* to E_p and $\theta_p^*=\theta_{-p}$ are defined by the following conditions;

(2.2) $$\langle x, \xi\rangle=(\theta_p x, \xi)_p=(x, \theta_p^*\xi)_{-p},\ \xi \in E_p,\ x \in E_p^*.$$

N. B. Generally we denote by the bracket $\langle\ ,\ \rangle$ the canonical bilinear form between a dual pair.

We are now ready to introduce a sequence of Fock spaces as follows. Let $E_p^{\hat{\otimes} n}$ be the n-fold symmetric tensor product and $e^{\hat{\otimes} E_p}$ be their direct sum with weight $\sqrt{n!}$; that is, $\Xi=(f_0, f_1, \cdots, f_n, \cdots) \in e^{\hat{\otimes} E_p}$, $\pi^n \Xi \equiv f_n \in E_p^{\hat{\otimes} n}$, has Hilbert norm

(2.3) $$\|\Xi\|^2_{e^{\hat{\otimes} E_p}}=\sum_{n=0}^{\infty} n!\,\|f_n\|^2_{E_p^{\hat{\otimes} n}}.$$

Then the injection $\iota_{p,q}$, induces injections $\iota_{p,q}^{\hat{\otimes} n}$ from $E_q^{\hat{\otimes} n}$ to $E_p^{\hat{\otimes} n}$ and $e^{\hat{\otimes}\iota_{p,q}}$ from $e^{\hat{\otimes} E_q}$ to $e^{\hat{\otimes} E_p}$, naturally. By these injections, we get a system of Hilbert spaces $\{e^{\hat{\otimes} E_p}; p \in Z\}$ such that $e^{\hat{\otimes} E_{-p}}$ is the dual of $e^{\hat{\otimes} E_p}$. Denote the projective limit and the inductive limit of the system $\{(e^{\hat{\otimes} E_p}, e^{\hat{\otimes}\iota_{p,q}});\ p, q \in Z\}$ by $e^{\hat{\otimes}\mathcal{E}}$ and $e^{\hat{\otimes}\mathcal{E}^*}$, respectively. Then $e^{\hat{\otimes}\mathcal{E}^*}$ is the dual of $e^{\hat{\otimes}\mathcal{E}}$.

For a given ξ in E_p, define an element $e^{\hat{\otimes}\xi}$ in $e^{\hat{\otimes} E_p}$ by

(2.4) $$e^{\hat{\otimes}\xi}\equiv(1, \xi, \xi^{\hat{\otimes} 2}/2!, \cdots, \xi^{\hat{\otimes} n}/n!, \cdots).$$

Then we have for $\xi, \eta \in E_p$, $x \in E_{-p}$

(2.5) $$(e^{\hat{\otimes}\eta}, e^{\hat{\otimes}\xi})_{e^{\hat{\otimes} E_p}}=e^{(\eta,\xi)_p} \quad \text{and} \quad \langle e^{\hat{\otimes} x}, e^{\hat{\otimes}\xi}\rangle=e^{\langle x,\xi\rangle}.$$

Theorem 2.1. (i) *If the injection $\iota_{p,q}$ is of Hilbert-Schmidt type and has norm $\|\iota_{p,q}\|_{H-S}<1$, so is $e^{\hat{\otimes}\iota_{p,q}}$ and its norm is dominated by $(1-\|\iota\|^2_{H-S})^{-1/2}$.*

(ii) *If $\mathcal{E}$ is a nuclear space, so is $e^{\hat{\otimes}\mathcal{E}}$.*

For a fixed p, define a symmetric tensor product $f_m \hat{\otimes} g_n$ of $f_m \in E_p^{\hat{\otimes} m}$ and $g_n \in E_p^{\hat{\otimes} n}$ by the symmetrization of the tensor product $f_m \otimes g_n \in E_p^{\otimes(m+n)}$. Let f_n be in $E_p^{\hat{\otimes} n}$ and G_k be in $E_{-p}^{\hat{\otimes} k}$, $n \geq k \geq 0$. Then $\langle G_k \hat{\otimes} F_{n-k}, f_n\rangle$ is a continuous linear functional of $F_{n-k} \in E_{-p}^{\hat{\otimes}(n-k)}$. There exists an element of $E_p^{\hat{\otimes}(n-k)}$, denote it by $G_k * f_n$, such that

(2.6) $$\langle F_{n-k}, G_k * f_n\rangle=\langle G_k \hat{\otimes} F_{n-k}, f_n\rangle.$$

Lemma 2.2. *For $g_k \in E_p^{\hat{\otimes} k}$, $f_{n-k} \in E_p^{\hat{\otimes}(n-k)}$, $f_n \in E_p^{\hat{\otimes} n}$ and $G_k \in E_{-p}^{\hat{\otimes} k}$,*

$$\|g_k \hat{\otimes} f_{n-k}\|_{E_p^{\hat{\otimes} n}} \leq \|g_k\|_{E_p^{\hat{\otimes} k}}\, \|f_{n-k}\|_{E_p^{\hat{\otimes}(n-k)}},$$

$$\|G_k * f_n\|_{E_p^{\hat{\otimes}(n-k)}} \leq \|G_k\|_{E_{-p}^{\hat{\otimes} k}}\, \|f_n\|_{E_p^{\hat{\otimes} n}}.$$

Define the following operators on $e^{\hat{\otimes} E_p}$ for $g_k \in E_p^{\hat{\otimes} k}$ and $G_k \in E_{-p}^{\hat{\otimes} k}$:

(2.7) $$a(G_k)\Xi \equiv \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} G_k * \pi^n \Xi,$$

(2.8) $$a^*(g_k)\Xi \equiv \sum_{n=0}^{\infty} g_k \hat{\otimes} \pi^n \Xi, \qquad \text{for } \Xi \in e^{\hat{\otimes} E_p}.$$

Theorem 2.3. *For $g_k \in E_p^{\hat{\otimes} k}$ and $G_k \in E_{-p}^{\hat{\otimes} k}$, we have*

(i) $\|a(G_k) e^{\hat{\otimes}\iota_{p,p+1}}\|_{L^2(e^{\hat{\otimes} E_{p+1}} \to e^{\hat{\otimes} E_p})} \leq \|G_k\|_{E_{-p}^{\hat{\otimes} k}} (1-\rho^2)^{-(k+1)/2} \rho^k (k!)^{1/2}$,

$$\|a^*(g_k)e^{\hat{\otimes}\iota_{p,p+1}}\|_{L^2(e^{\hat{\otimes}E_{p+1}}\to e^{\hat{\otimes}E_p})}\leq\|g_k\|_{E_p^{\hat{\otimes}k}}(1-\rho^2)^{-(k+1)/2}(k!)^{1/2}.$$

(ii) $\langle\Psi, a(G_k)\Xi\rangle=\langle a^*(G_k)\Psi, \Xi\rangle$,

for $\Psi\in e^{\hat{\otimes}\iota_{-p,-p+1}}e^{\hat{\otimes}E_{-p+1}}\subset e^{\hat{\otimes}E_{-p}}$ *and* $\Xi\in e^{\hat{\otimes}\iota_{p,p+1}}e^{\hat{\otimes}E_{p+1}}\subset e^{\hat{\otimes}E_p}$.

Theorem 2.4. *For* $f_m\in E_p^{\hat{\otimes}m}$, $g_n\in E_p^{\hat{\otimes}n}$, $f\in E_p$ *and* $G\in E_{-p}$, *we have*

$$a(f_m)a(g_n)=a(f_m\hat{\otimes}g_n),\qquad a^*(f_m)a^*(g_n)=a^*(f_m\hat{\otimes}g_n)$$

and

$$a(G)a^*(f)-a^*(f)a(G)=\langle G, f\rangle.$$

§3. Hilbert spaces of non-linear functionals on $\mathcal{E}$. Denote by $\mathcal{K}_0$ the linear combination of non-linear functionals $\{e^{\langle\iota_{-\infty,\infty}\eta,\xi\rangle}; \eta\in\mathcal{E}\}$ in $\xi\in\mathcal{E}=E_\infty$. We introduce a sequence of inner products such that

(3.1) $$(e^{\langle\iota_{-\infty,\infty}\eta,\xi\rangle}, e^{\langle\iota_{-\infty,\infty}\zeta,\xi\rangle})^{(p)}\equiv e^{(\iota_{p,\infty}\eta,\iota_{p,\infty}\zeta)_p}.$$

Denote by $\mathcal{F}^{(p)}$ the completion of $\mathcal{K}_0$ with respect to $\|\ \|^{(p)}$. Then $\mathcal{F}^{(p)}$ is a space of continuous non-linear functionals on $\mathcal{E}$ and the inclusions

(3.2) $$\mathcal{F}^{(p+1)}\subset\mathcal{F}^{(p)},\qquad p\in Z,$$

hold. Let $\mathcal{F}=\mathcal{F}^{(\infty)}$ be the projective limit of $\mathcal{F}^{(p)}$ and $\mathcal{F}^*=\mathcal{F}^{(-\infty)}$ be the inductive limit of $\mathcal{F}^{(p)}$.

Theorem 3.1. *For* $\infty\geq p\geq-\infty$, $\mathcal{F}^{(p)}$ *is isomorphic to* $e^{\hat{\otimes}E_p}$ *by the isomorphism* $\theta^{\mathcal{E}\to\mathcal{F}}$

$$\theta^{\mathcal{E}\to\mathcal{F}}: \Xi\longmapsto\theta^{E\to F}(\Xi)=\langle e^{\hat{\otimes}\iota_{-p,\infty}\xi}, \Xi\rangle.$$

Remark 3.2. Let $\mathcal{K}(E_p)$ be the Hilbert space with the *reproducing kernel* $e^{(\eta,\xi)_p}$, $\infty>p>-\infty$ (see Aronszajn [6], Hida-Ikeda [5]). Then the map $\theta^{\mathcal{K}\to\mathcal{F}}$ from $\mathcal{K}(E_{-p})$ to $\mathcal{F}^{(p)}$;

(3.3) $$U\longmapsto\theta^{\mathcal{K}\to\mathcal{F}}(U)=U(\iota_{-p,\infty}\xi),$$

is one-to-one onto linear. In other words, $U(\xi)$ in $\mathcal{F}^{(p)}$ can be extended to a continuous functional $U_p(x)$ on E_{-p} and $U_p(x)$ is in $\mathcal{K}(E_{-p})$.

A non-linear functional $U(\xi)$ on $E=E_\infty$ is *n-times E_p-Fréchet differentiable* if there exist k-ple symmetric linear forms $U^{(k)}(\xi; \eta_1, \cdots, \eta_k)$ for $1\leq k\leq n$, satisfying the following (3.4) and (3.5);

(3.4) $$\left|U(\xi+\eta)-U(\xi)-\sum_{k=1}^{n}\frac{1}{k!}U^{(k)}(\xi; \eta, \eta, \cdots, \eta)\right|=o(\|\iota_{p,\infty}\eta\|_p^n),$$

(3.5) $$|U^{(k)}(\xi; \eta_1, \cdots, \eta_k)|\leq\text{const.}\ \|\iota_{p,\infty}\eta_1\|_p\cdots\|\iota_{p,\infty}\eta_k\|_p,\qquad 1\leq k\leq n.$$

Then $U^{(n)}(\xi; \eta_1, \cdots, \eta_n)$ is called *Fréchet derivative of $U(\xi)$ of order n*. If $U(\xi)$ is n-times E_p-Fréchet differentiable, then $U^{(n)}(\xi; \eta_1, \cdots, \eta_n)$ can be regarded as a continuous n-ple symmetric linear form on E_p.

Theorem 3.3. *If $U(\xi)$ is in $\mathcal{F}^{(p)}$, then*

(i) *$U(\xi)$ is arbitrary times E_{-p}-Fréchet differentiable and*

$$U(\xi+\eta)=U(\xi)+\sum_{n=1}^{\infty}\frac{1}{n!}U^{(n)}(\xi; \eta, \cdots, \eta).$$

(ii) *There exists* $\Xi\in{}^{\hat{\otimes}E_p}$ *and* $U^{(n)}$ *can be extended to a linear functional on* $E_{-p+1}^{\hat{\otimes}n}$, *in such way that for* $F_n\in E_{-p+1}^{\hat{\otimes}n}$

$$U^{(n)}(\xi; F_n)=\langle F_n\hat{\otimes}e^{\hat{\otimes}\iota_{-p,\infty}\xi}, \Xi\rangle$$

$$=\langle e^{\hat{\otimes}\iota_{-p+1,\infty}\xi},\ a(F_n)e^{\hat{\otimes}\iota_{p-1,p}\Xi}\rangle.$$

(iii) *For fixed* $\eta_1, \cdots, \eta_n \in E_{-p}$, *the mapping from* $\mathcal{F}^{(p)}$ *to* $\mathcal{F}^{(p-1)}$:

$$U(\xi)\longmapsto U^{(n)}(\xi\,;\eta_1,\cdots,\eta_n) \quad \text{is continuous.}$$

§ 4. **Traceable space E.** In what follows, we will treat only Hilbert spaces of functions which are naturally imbedded into L^2 spaces and their duals. Therefore, for simplicity, we can omit the notations of injections without confusions. Let T be a separable metrizable space with a σ-finite Borel measure ν. Let E be a dense linear subset of $L^2(T,\nu)$ which is itself a Hilbert space with inner product $(\ ,\ )_E$ with $\|\xi\|_E \geq \|\xi\|_{L^2(T,\nu)}$.

Definition 4.1. The space E is called *traceable* if the linear functional $\delta_t : \xi \rightarrow \xi(t) \in R$ for $\xi \in E$ is well defined in E^* and if the mapping $t \mapsto \delta_t \in E^*$ is strongly continuous in $t \in T$.

If E is traceable, then every element ξ of E is continuous on T and so is $f_n \in E^{\hat{\otimes} n}$ on T^n. For any $f_n \in E^{\hat{\otimes} n}$, the mapping $t \rightarrow \delta_t * f_n$ from T to $E^{\hat{\otimes}(n-1)}$ is continuous in t. Further

$$\delta_{t_n} * \cdots * \delta_{t_1} * f_n = f_n(t_1, \cdots, t_n) \tag{4.1}$$

is a continuous function belonging to $\hat{L}^2(T^n, \nu^n)$ and this realized the injection $\iota^{\hat{\otimes} n}$. A dual element $F_n \in E^{*\hat{\otimes} n}$ is not necessarily a function on T^n, but it is convenient to write $F_n(u_1, \cdots, u_n)$ as if a function on T^n.

Lemma 4.2. *Let E be traceable. Then the injection ι from E into $L^2(T,\nu)$ satisfies*

$$\|\delta\|^2 \equiv \int_T \|\delta_t\|_{E^*}^2 d\nu(t) = \|\iota\|_{H-S}^2.$$

Lemma 4.3. *Assume that $\|\delta\|^2 < \infty$. If f is in $E^{\hat{\otimes} 2}$, then $f(t,t) = \delta_t * \delta_t f$ is integrable and*

$$\int_T f(t,t)d\nu(t) = \sum_k \langle \eta_k \otimes \eta_k, f\rangle$$

holds for any c.o.n.s. $\{\eta_k\}$ in $L^2(T,\nu)$. Furthermore, there exists an o.n.s. $\{\eta_k^f\}$ in $L^2(T,\nu)$ such that $\eta_k^f \in E$ and that

$$f = \sum_{k=1}^{\infty} \rho_k \eta_k^f \hat{\otimes} \eta_k^f \quad \textit{with} \quad \sum_k |\rho_k| < \infty.$$

Since $\delta_t \in E^*$, $a_t \equiv a(\delta_t)$ is an operator on $e^{\hat{\otimes} E}$ and $a_t^* \equiv a^*(\delta_t)$ is an operator on $e^{\hat{\otimes} E^*}$. If $U(\xi)$ is E^*-differentiable, then $U'(\xi\,;\eta)$ can be extended to a bounded linear functional on E^* for fixed $\xi \in E$. Furthermore $U'(\xi\,;t) \equiv U'(\xi : \delta_t)$ is a function on T, which belongs to E.

We now return to the setup in § 3. We assume that E_0 is equal to $L^2(T,\nu)$ and that $\{E_p\}_{p\in Z}$ are given as in § 1.

Theorem 4.4. *Suppose that the injection $\iota_{0,1} : E_1 \rightarrow L^2(T,\nu)$ is traceable. Then*

(i) *the functional derivative*

$$\frac{\delta}{\delta\xi(t)} : U(\xi) \longmapsto U'(\xi\,;t)$$

is a continuous operator on $\mathcal{F}$ and is strongly continuous in t. Specially, if $U(\xi)=\langle \Xi, e^{\hat{\otimes}\xi}\rangle$ with $\Xi \in e^{\hat{\otimes}E}$, then

$$U'(\xi\,;t)=\langle a_t\Xi, e^{\hat{\otimes}\xi}\rangle=\sum_{n=1}^{\infty} n\langle \delta_t*\pi^n\Xi, \xi^{\hat{\otimes}(n-1)}\rangle. \tag{4.2}$$

(ii) *The multiplication*

$$\xi(t)\cdot\ : U(\xi)\longmapsto \xi(t)U(\xi)$$

is a continuous operator on $\mathcal{F}^*$ and strongly continuous in t.

Remark 4.5. If $U(\xi)$ is in $\mathcal{F}^{(p)}$, with $p\geq 1$, then $U^{(k)}(\xi\,;t_1,\cdots,t_k)$ is in $E_p^{\hat{\otimes}k}$. For a given $U(\xi)\in\mathcal{E}^{(-p)}$ and a fixed $\xi\in\mathcal{E}$, $U^{(k)}(\xi\,;\eta_1,\cdots,\eta_k)$ is a continuous multi-linear functional on E_p, and hence we can define $\delta/\delta\xi(t_1)\cdots\delta/\delta\xi(t_k)$ as an operator valued generalized function in $(t_1,\cdots,t_k)$. In particular, if $U(\xi)$ is in $\mathcal{F}^{(0)}$, then there exists an L^2-function $U^{(k)}(\xi\,;t_1,\cdots,t_k)=(\delta/\delta\xi(t_1)\cdots\delta/\delta\xi(t_k))U(\xi)$, such that

$$U^{(k)}(\xi\,;\eta_1,\cdots,\eta_k)=\int_{T^k} U^{(k)}(\xi\,;t_1,\cdots,t_k)\eta_1(t_1)\cdots\eta_k(t_k)d\nu^k. \tag{4.3}$$

References

[1] Hida, T.: Analysis of Brownian functionals. Carleton Math. Lect. Notes, no. 13, 2nd ed. (1978).

[2] ——: Brownian motion. Springer Verlag A.M., vol. 11 (1980).

[3] Segal, I.: Tensor algebras over Hilbert spaces. I. Trans. Amer. Math. Soc., **81**, 106–134 (1956).

[4] ——: Tensor algebras over Hilbert spaces. II. Ann. of Math., **63**, 160–175 (1956).

[5] Hida, T., and Ikeda, N.: Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral. Proc. Fifth Berkeley Symp. on Math. Statist. and Probability, vol. 2, part 1, pp. 117–143 (1967).

[6] Aronszajn, N.: Theory of reproducing kernel. Trans. Amer. Math. Soc., **68**, 337–404 (1950).