87. Characteristic Cauchy Problems and Solutions of Formal Power Series

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(Communicated by Kôsaku YOSIDA, M. J. A., Oct. 13, 1980)

§1. **Introduction.** Let C^{n+1} be the $(n+1)$-dimensional complex space. $z=(z_0, z')=(z_0, z_1, \cdots, z_n)$ denotes its point and $\xi=(\xi_0, \xi')=(\xi_0, \xi_1, \cdots, \xi_n)$ denotes its dual variable. We shall make use of the notation $\partial_z=(\partial_{z_0}, \partial_{z'})=(\partial_{z_0}, \partial_{z_1}, \cdots, \partial_{z_n})$, $\partial_{z_i}=\partial/\partial z_i$. For a linear partial differential operator $a(z, \partial_z)$, $a(z, \xi)$ denotes its total symbol. Now let us consider Cauchy problem in a neighbourhood Ω of $z=0$,

$$\text{(C.P)} \qquad \begin{cases} L(z, \partial_z)u(z)=((\partial_{z_0})^k - A(z, \partial_z))u(z)=f(z), \\ (\partial_{z_0})^i u(0, z')=\hat{u}_i(z'), \qquad 0\le i\le k-1, \end{cases}$$

where

$$(1.1) \qquad A(z, \partial_z)=\sum_{i=0}^{k-1} A_i(z, \partial_{z'})(\partial_{z_0})^i$$

and $A(z, \partial_z)$ is an operator of order m and its coefficients and $f(z)$ are holomorphic in Ω and $\hat{u}_i(z')$ $(0\le i\le k-1)$ are holomorphic in $\Omega'=\Omega\cap\{z_0=0\}$. We can easily find out a solution of formal power series $\hat{u}(z)$ of (C.P) of the form

$$(1.2) \qquad \hat{u}(z)=\sum_{n=0}^{\infty} \hat{u}_n(z')(z_0)^n/n!.$$

$\hat{u}_n(z')$ $(n\ge k)$ are successively and uniquely determined from (C.P). It follows from well-known Cauchy-Kovalevskaja theorem that whenever $m\le k$, $\hat{u}(z)$ converges and is a unique holomorphic solution of (C.P).

The purpose of this paper is to give an anlytical interpretation of $\hat{u}(z)$, that is, existence of a solution $u_S(z)$ of the equation $L(z, \partial_z)u_S(z)=f(z)$ with the asymptotic expansion $\hat{u}(z)$ in a sector S, when $m>k$. So we assume $m>k$ in the following.

§2. **Characteristic indices.** In §2 we introduce a new notation, characteristic indices. Let us write $A(z, \partial_z)$ in the form different from (1.1),

$$(2.1) \qquad A(z, \partial_z)=\sum_{i=0}^{m}\left(\sum_{l=s_i}^{i} a_{i,l}(z, \partial_{z'})(\partial_{z_0})^{i-l}\right),$$

where $a_{i,l}(z, \xi')$ is a homogeneous polynomial of ξ' with degree l and if $a_{i,l}(z, \xi')\equiv 0$ for all l, we put $s_i=+\infty$. We expand $a_{i,l}(z, \xi')$ at $z_0=0$ with respect to z_0,

$$(2.2) \qquad a_{i,l}(z, \xi')=\sum_{j=0}^{\infty} a_{i,l,j}(z', \xi')(z_0)^j.$$

Put

$$(2.3)\qquad \begin{cases} d_i=\min\{(l+j)\,;\,a_{i,l,j}(z',\xi')\not\equiv 0\} & (i>k): \\ d_k=1. \end{cases}$$

If $s_i=+\infty$, we put $d_i=+\infty$.

Let us define quantities σ_i $(0\leq i\leq l)$. Consider the set $P=\{P_j=(j,d_j)\,;\,k\leq j\leq m\}$ in R^2. Let $\hat{P}$ be the convex envelope of the set P. The lower convex part of the boundary of $\hat{P}$ consists segments $\sum_i$ $(1\leq i\leq l)$ (see Fig. 2.1).

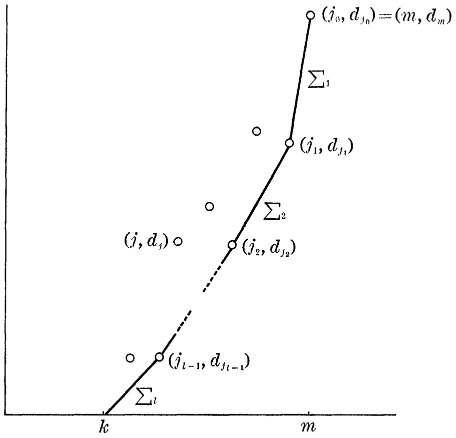

Fig. 2.1

We denote Δ the set of extremal points (vertexes) of $\sum_i$ $(1\leq i\leq l)$. Put $\Delta=\{(j_i, d_{j_i})\,;\,i=0,1,\cdots,l\}$, where $m=j_0>j_1>\cdots>j_l=k$.

Definition 2.1. The i-th characteristic index σ_i is defined by

$$(2.4)\qquad \begin{cases} \sigma_0=+\infty, \\ \sigma_i=(d_{j_{i-1}}-d_{j_i})/(j_{i-1}-j_i) & \text{for } i=1,2,\cdots,l. \end{cases}$$

From the definition $+\infty=\sigma_0>\sigma_1>\sigma_2\cdots>\sigma_l>1$.

Remark 2.2. σ_1 is a generalization of the irregularity of characteristic elements in Komatsu [1]. Characteristic indices can be defined for more general operators.

§3. Theorems. In order to state theorems, we consider functions of several complex variables with an asymptotic expansion with respect to one of them. Put $S=S(a,b)=\{z_0\in C^1\,;\,a<\arg z_0<b\}$, $U=\{z\in C^{n+1}\,;\,|z_0|<r_0, |z_i|<r\ (1\leq i\leq n)\}$, $U'=\{z'\in C^n\,;\,|z_i|<r\}$ and $U_S=(\{|z_0|<r_0\}\cap S)\times U'$.

Definition 3.1. Let $f(z)$ be holomorphic in U_S. A formal power series

$$(3.1)\qquad \sum_{n=0}^{\infty} a_n(z')(z_0)^n/n!,$$

where $a_n(z')$ $(n=0,1,\cdots)$ are holomorphic in U', is said to represent $f(z)$ asymptotically in U_S, if for any N

(3.2) $$|z_0|^{-N}\left|f(z)-\sum_{n=0}^{N}a_n(z')(z_0)^n/n!\right|$$

tends to zero uniformly on any compact set in U' as z_0 tends to zero in S.

The asymptotic relationship of the definition is usually written in the form

(3.3) $$f(z)\sim\sum_{n=0}^{\infty}a_n(z')(z_0)^n/n! \qquad \text{as } z_0\to 0 \text{ in } U_S.$$

By $\tilde{\mathcal{O}}(U-\{z_0=0\})$ we denote the set of holomorphic functions on the universal covering space of $U-\{z_0=0\}$ and by $\tilde{\mathcal{O}}(U\times(|\lambda|>\Lambda))$ the set of holomorphic functions of $(n+2)$-variables (z,λ) on the covering space of $U\times(|\lambda|>\Lambda)$. By $C(d,\theta)$ $(d>0)$ simply $C(\theta)$ we denote a path in λ-space, which starts at $\infty\exp(i(-\pi+\theta))$, goes to $d\exp(i(-\pi+\theta))$ straightly, goes around the origin once on $|\lambda|=d$ and ends at $\infty\exp(i(\pi+\theta))$ (see Fig. 3.1).

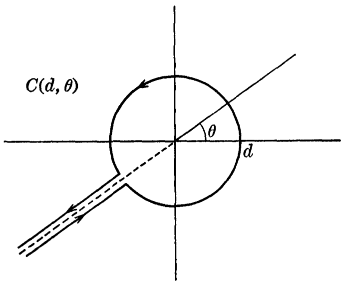

Fig. 3.1

Theorem 3.2. *Let $S=S(a,b)$ be a sector with $(b-a)<\pi/(\sigma_l-1)$ and $(\pi+b-a)/2<\theta_1<(\pi\gamma_1)/2$, $\gamma_1=\sigma_l/(\sigma_l-1)$. Then there are functions $u_{0,S}(z)$, $g_{1,S}(z)\in\tilde{\mathcal{O}}(U-\{z_0=0\})$ in a neighbourhood U of $z=0$, $U\subset\Omega$, such that*

(3.4) $$\begin{cases} L(z,\partial_z)u_{0,S}(z)=f(z)+g_{1,S}(z),\\ u_{0,S}(z)\sim\hat{u}(z) & \text{as } z_0\to 0 \text{ in } U_S,\\ g_{1,S}(z)\sim 0 & \text{as } z_0\to 0 \text{ in } U_S.\end{cases}$$

Here $g_{1,S}(z)$ is represented in the form, if $|\arg z_0+\theta|<\pi/2$,

(3.5) $$g_{1,S}(z)=\frac{1}{2\pi i}\int_{C(\theta)}\exp(\lambda z_0)G_{1,S}(z,\lambda)d\lambda,$$

where $G_{1,S}(z,\lambda)\in\tilde{\mathcal{O}}(U\times(|\lambda|>\Lambda))$ and satisfies

(3.6) $$\sup_{z\in U}|G_{1,S}(z,\lambda)|\le A\exp(c'|\lambda|^{1/\gamma_1})$$

and if $|\arg\lambda+(a+b)/2|<\theta_1$,

(3.7) $$\sup_{z\in U}|G_{1,S}(z,\lambda)|\le A\exp(-c|\lambda|^{1/\gamma_1}).$$

Λ, A, c' and c are positive constants.

Remark 3.3. It follows from well-known Borel-Ritt theorem for asymptotic series that there exist $u_{0,S}(z)$ and $g_{1,S}(z)$ satisfying (3.4), but we do not use it. It is important in Theorem 3.2 that $g_{1,S}(z)$ is represented in the form (3.5) by $G_{1,S}(z,\lambda)$ with estimates (3.6) and (3.7).

Now let us cancel $g_{1,S}(z)$. To do so we put a sufficient condition on $L(z,\partial_z)$:

Condition 1. For $(i,d_i)\in\Delta$ $(i>k)$, $d_i=s_i$ and

(3.8) $$\prod_{\substack{(i,s_i)\in\Delta\\ i>k}} a_{i,s_i}(0,\xi')\not\equiv 0.$$

Theorem 3.4. *Suppose that $L(z,\partial_z)$ satisfies Condition 1. Let $S=S(a,b)$ be a sector with $(b-a)<\pi/(\sigma_1-1)$. Then there is a function $u_S(z)\in\tilde{\mathcal{O}}(U-\{z_0=0\})$ in a neighbourhood U of $z=0$ such that*

(3.9) $$\begin{cases} L(z,\partial_z)u_S(z)=f(z),\\ u_S(z)\sim\hat{u}(z) \qquad \text{as } z_0\to 0 \text{ in } U_S.\end{cases}$$

Let us give an application of Theorem 3.4. Let us regard the operator $L(z,\partial_z)$ as an operator $L(x,\partial_x)$ with analytic coefficients on a domain $\Omega_R=\Omega\cap\{\mathrm{Im}\, z=0\}$ in R^{n+1} by the restriction. We denote x by the point in R^{n+1}. We consider Cauchy problem in Ω_R,

$(\mathrm{C.P})_R$ $$\begin{cases} L(x,\partial_x)u(x)=\{(\partial_{x_0})^k-A(x,\partial_x)\}u(x)=f(x),\\ (\partial_{x_0})^i u(0,x')=u_i(x'), \qquad 0\le i\le k-1.\end{cases}$$

In general, $(\mathrm{C.P})_R$ is not solvable. But we have

Theorem 3.5. *Suppose that $L(x,\partial_x)$ satisfies Condition 1 and $f(x)$ and $u_i(x')$ $(0\le i\le k-1)$ are analytic in x and x' respectively in a neighbourhood of the origin. Then $(\mathrm{C.P})_R$ has a solution $u(x)$ in a neighbourhood V of $x=0$, which is C^∞ in V and analytic in $V-\{x_0=0\}$. Moreover we have*

(3.10) $$|(\partial_{x_0})^{\alpha_0}(\partial_{x'})^{\alpha'}u(x)|\le AC^{|\alpha|}(\alpha_0!)^{\gamma_1}(\alpha'!) \qquad \text{for } x\in V,$$

where $\gamma_1=\sigma_l/(\sigma_l-1)$, α denotes multi-indices and A and C are constants.

In order to construct the functions $u_{0,S}(z)$ and $u_S(z)$, we make full use of functions in the form

(3.11) $$\frac{1}{2\pi i}\int_{C(\theta)} \exp(\lambda z_0)V(z,\lambda)d\lambda$$

and investigate equations with a parameter λ. This method of construction of functions is slight similar to that used in Ōuchi [2], [3]. The details and proofs will be published elsewhere.

References

[1] Komatsu, H.: Irregularity of characteristic elements and construction of null solutions. J. Fac. Sci. Univ. Tokyo, **23**, 297–342 (1976).

[2] Ōuchi, S.: Asymptotic behaviour of singular solutions of linear partial differential equations in the complex domain. Ibid., **27**, 1–36 (1980).

[3] ——: An integral representation of singular solutions of linear partial differential equations in the complex domain. Ibid., **27**, 37–85 (1980).