9. On Unramified $SL_2(F_4)$ Extensions of an Algebraic Function Field

By Hidenori KATSURADA
Department of Mathematics, Hokkaido University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 12, 1980)

The purpose of this note is to report some results on the number of unramified $SL_2(F_4)$ extensions of some algebraic function field of characteristic 2. Detailed accounts are stated in [1] and [2].

§0. **Main results.** Let k be an algebraically closed field of characteristic 2. Let $K=k(x, y)$ be an algebraic function field over k defined by $y^2-y=x^5-\alpha x^3$ $(\alpha\in k)$. Let $\tilde{K}$ be the maximum unramified Galois extension of K and let $A_{SL_2(F_4)}$ be the set of $GL_2(k)$ equivalence classes of representations of $\mathrm{Gal}(\tilde{K}/K)$ onto $SL_2(F_4)$. We put

$$B=\left\{\begin{array}{ll}(X, Y, Z, \lambda)\in P^2\times A^1; & X^2Z^2+Y^3Z+(c_4X+Y)X^3=0,\\ & Y^9Z^8+ZX^{16}+c_4^2YX^{16}\\ & \qquad +(X+\alpha^2Y)(Y^8X^8+\alpha^4X^{16})=0,\\ & YX^{16}+(X+\alpha^2Y)(X^{16}\alpha^8+Y^{16})=0,\\ & c_4=\lambda^{16}+\alpha^4\lambda^8+\alpha^2\lambda^2+\lambda,\ Z\neq 0,\\ & \alpha^2Z^2Y+Y^2Zc_4+X^3c_4\neq 0\end{array}\right\}.$$

Then one of our main results is:

Theorem 1. *There is a* 2 : 1 *map of* B *onto* $A_{SL_2(F_4)}$.

By making use of this theorem and some other considerations, we can show the following

Theorem 2. $\sharp A_{SL_2(F_4)}=640$ *if* $\alpha=0$,
$=736$ *otherwise.*

Corollary to Theorem 2. *The number of unramified* $SL_2(F_4)$ *extensions of* K *is* 320 *if* $\alpha=0$ *and* 368 *otherwise.*

§1. **Representations of** $\mathrm{Gal}(\tilde{K}/K)$ **into** $GL_n(F_q)$. Let K_A be the adele ring of K, let $\mathfrak{O}$ be the integer ring, and let $\mathfrak{U}$ be the unit group of $\mathfrak{O}$. We put $G_n=GL_n(\mathfrak{O})\backslash GL_n(K_A)/GL_n(K)$. Then, the map $GL_n(K_A)\ni(u_{ij})\mapsto(u_{ij}^q)\in GL_n(K_A)$ induces a map $F(q)$ of G_n into itself. We denote by $\mathrm{Rep}(GL_n(F_q))$ the set of $GL_n(k)$ equivalence classes of representations of $\mathrm{Gal}(\tilde{K}/K)$ into $GL_n(F_q)$. Then we have:

Proposition 1.1. *There is a one to one correspondence between the set* $G_n^{F(q)}$ *of* $F(q)$ *fixed points of* G_n *and* $\mathrm{Rep}(GL_n(F_q))$.

For any element R of $GL_n(K_A)$, we denote by $[R]$ the element of G_n whose representative is R.

Corollary to Proposition 1.1. *We put*

$$S_n=\{[R]\in G_n \text{ satisfying } \det R=1\}.$$

Then there is a one to one correspondence between $S_n^{F(q)}$ *and* $\mathrm{Rep}\,(SL_n(F_q))$.

Though our result is written in the terminology of adeles, the above proposition is essentially equivalent to the result in [3].

Definition 1.2. Let R be an element of $SL_2(K_A)$. Then an element r_0 of K_A^* is said to be a maximal element of R if $[R]=\left[\begin{pmatrix} r_0 & s_0 \\ 0 & r_0^{-1}\end{pmatrix}\right]$ and $\deg r_0 \geqq \deg r$ for any element r of K_A^* satisfying $[R]=\left[\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix}\right]$. We choose and fix a maximal element of R and denote it by max R.

Definition 1.3. An element R of $SL_2(K_A)$ is semi-stable (resp stable) if the degree of max R is not positive (resp negative). We also say an element of S_2 is semi-stable (resp stable) if its representative is semi-stable (resp stable).

Proposition 1.4. *An element* $[R]$ *of* $S_2^{F(q)}$ *is stable if and only if* $[R]$ *corresponds to an irreducible representation.*

For a ring A, we put

$$T_n(A)=\{u=(u_{ij}) \in GL_n(A)\,;\ u_{ij}=0 \text{ if } i<j\}.$$

We introduce an equivalence relation on $GL_n(K_A)$: For any two elements a, b of $GL_n(K_A)$, $a \approx b$ if and only if there are elements $u \in T_n(\mathfrak{O})$ and $v \in T_n(K)$ satisfying $b=uav$.

We need the following propositions to describe $\mathrm{Rep}\,(SL_2(F_4))$.

Proposition 1.5. *Let* r *be an element of* K_A^*. *Let* $R_i=\begin{pmatrix} r & s_i \\ 0 & r^{-1}\end{pmatrix}$ $(i=1,2)$ *be two elements of* $SL_2(K_A)$ *which have* r *as a maximal element. Then,* $R_1 \approx R_2$ *if and only if* $[R_1]=[R_2]$.

Proposition 1.6. *Let the genus of* K *be* 2. *Then, for every stable element* R *of* $SL_2(K_A)$, $[R]=\left[\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix}\right]$ *with an element* r *of* K_A^* *corresponding to* P^{-1} *for some prime* P *of* K.

§2. **The structure of $B_r(q)$.** Now let r be an element of K_A^* corresponding to a divisor of the form P^{-1} with a prime divisor P of K. We put

$$B_r(q)=\left\{R=\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix};\ [R] \in S_2^{F(q)}\right\}\Big/\approx$$

$$B_r'(q)=\left\{\begin{array}{l} R=\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix};\ \text{there are non zero elements } u \in M_2(\mathfrak{O}) \text{ and} \\ \qquad\qquad v \in M_2(K) \text{ satisfying } R^{(q)}v=uR \end{array}\right\}\Big/\approx.$$

We assume that the genus of K is 2. Then it follows from Proposition 1.6 that every stable element u of S_2 can be expressed as $u=\left[\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix}\right]$ with some element r. Hence noting Corollary to Propositions 1.1 and 1.4, to study irreducible representations of $\mathrm{Gal}\,(\tilde{K}/K)$

into $SL_2(F_q)$, first we must study $B_r(q)$. The main result of this section is:

Proposition 2.1. *For any r and q, there are $q+1$ polynomials h, $\{f_i\}_{1\leq i\leq q}$ of $k[X_1, X_2, X_3, Y_1, \cdots, Y_q]$ and $q+3$ polynomials $\{g_i\}_{1\leq i\leq q+3}$ of $k[X_1, X_2, X_3, Y_1, \cdots, Y_q, Z_1, \cdots, Z_{q-2}]$ satisfying the following conditions: There is a one to one correspondence between $B'_r(q)$ and the set*

$$B_1=\left\{\begin{array}{ll}(a_1, a_2, a_3)\in P^2\,; & h(a_1, a_2, a_3, b_1, \cdots, b_q)=f_i(a_1, a_2, a_3, b_1, \cdots, b_q)=0, \\ & g_i(a_1, a_2, a_3, b_1, \cdots, b_q, u_1, \cdots, u_{q-2})=0 \quad 1\leqq i\leqq q+2 \\ & \text{for some } (b_1, \cdots, b_q, u_1, \cdots, u_{q-2}) \text{ of } P^{2q-3}\end{array}\right\}$$

and $B_r(q)$ is mapped bijectively to the following subset B_2 of B_1:

$$B_2=\{(a_1, a_2, a_3)\in B_1\,;\, g_{q+3}(a_1, a_2, a_3, b_1, \cdots, b_q, u_1, \cdots, u_{q-2})\neq 0\}.$$

Remark 2.2. If $q\geqq 4$, we can take $h=0$.

§3. Representations of Gal $(\tilde{K}/K)$ onto $SL_2(F_4)$. In this section, we characterize representations of Gal $(\tilde{K}/K)$ onto $SL_2(F_4)$. Let k be an algebraically closed field of characteristic 2. Let G be a finite group, and let ρ be a representation of G into $SL_2(k)$. Let V be a G-module associated with ρ. Let e_1, e_2 be a basis of V. We define elements $u_1, \cdots, u_{n+1}$ of $V^{\otimes n}$ by $u_1=e_1\otimes\cdots\otimes e_1$, $u_2=\sum_i e_1\otimes\cdots\otimes \underset{i}{e_2}\otimes\cdots\otimes e_1$, $u_3=\sum_{i,j} e_1\otimes\cdots\otimes \underset{i}{e_2}\cdots\otimes \underset{j}{e_2}\otimes\cdots\otimes e_1$, $\cdots$, $u_{n+1}=e_2\otimes\cdots\otimes e_2$. Then the vector space spanned by $u_1, u_2, \cdots, u_{n+1}$ is also a G-module.

Proposition 3.1. *Let G be a subgroup of $SL_2(F_4)$. Let ρ be a representation of G into $SL_2(k)$. Then $\rho(G)\cong SL_2(F_4)$ if and only if $V_5^G=0$ and $V_3^G=0$.*

We put

$$A_r(4)=\left\{R=\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix};\ [R]\in S_2^{F(4)} \text{ and the image of a representation corresponding to } R \text{ is isomorphic to } SL_2(F_4)\right\}.$$

Then, using the above proposition, we obtain:

Corollary to Proposition 3.1. *Let K be an algebraic function field of characteristic* 2 *and of genus* 2. *Let P be a non Weierstrass point of K, and let r be an element of K_A^* corresponding to P^{-1}.* Then,

$$A_r(4)=B_r(4)-(B_r(4)\cap B'_r(2)).$$

§4. The outline of the proof of Theorem 1. Let K be an algebraic function field stated in § 0. Then the genus of K is 2 and K has only one Weierstrass point P_∞ which is the extension of the denominator of (x) in $k(x)$ to K. First:

Proposition 4.1. *Let r be an element of K_A^* corresponding to P_∞^{-1}. Then if $[R]=\left[\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix}\right]$ is an element of $S_2^{F(4)}$, $[R]$ is not stable.*

Next let P_λ be a non Weierstrass point, and let r_λ be an element of K_A^* corresponding to P_λ^{-1}. Then, applying Proposition 2.1 and Corollary to Proposition 3.1 to this case, we obtain:

Proposition 4.2. *There is a one to one map ϕ_{r_λ} of $A_{r_\lambda}(4)$ to the set*

$$\left\{\begin{array}{l}(X, Y, Z) \in P^2;\ X^2Z^2+Y^3Z+(c_4X+Y)X^3 \\ \qquad\qquad =YX^{16}+(X+\alpha^2Y)(Y^{16}+X^{16}\alpha^8)=0, \\ \qquad\qquad Y^9Z^8+ZX^{16}+c_4^2YX^{16}+(X+\alpha^2Y)(Y^8X^8+\alpha^4X^{16})=0, \\ \qquad\qquad c_4=\lambda^{16}+\alpha^4\lambda^8+\alpha^2\lambda^2+\lambda,\ Z\neq 0,\ \alpha^2Z^2Y+Y^2Zc_4+X^3c_4\neq 0\end{array}\right\}.$$

To complete the proof, we need the following two lemmas:

Lemma 4.3. *Let P, P' be prime divisors of K which are extensions of a prime divisor Q of $k(x)$. Let r (resp r') be an element of K_A^* corresponding to P^{-1} (resp P'^{-1}). Let $R=\begin{pmatrix} r & s \\ 0 & r^{-1}\end{pmatrix}$ be a stable element. Then $[R]=\left[\begin{pmatrix} r' & s' \\ 0 & r'^{-1}\end{pmatrix}\right]$ with some s' of K_A.*

Lemma 4.4. *Let P_λ be a non Weierstrass point and let r_λ be an element of K_A^* corresponding to P_λ^{-1}. Let $R=\begin{pmatrix} r_\lambda & s \\ 0 & r_\lambda^{-1}\end{pmatrix}$ be a stable element and $[R] \in S_2^{F(4)}$. Then there is only one element μ of k different from λ satisfying $[R]=\left[\begin{pmatrix} r_\mu & s' \\ 0 & r_\mu^{-1}\end{pmatrix}\right]$.*

Proof of Theorem 1. For any element λ of k, there is a prime P satisfying $PP'P_\infty^{-2}=(x-\lambda)$. We choose and fix such a prime and denote it by P_λ. Now let (a_1, a_2, a_3, λ) be an element of B. Then there is a prime P_λ and $\phi_{r_\lambda}((a_1\cdot a_2\cdot a_3))$ is an element of $A_{r_\lambda}(4)$. Conversely let $[R]$ be an element of $A_{SL_2(F_4)}$. Then it follows from Proposition 1.5 that there is an element r of K_A^* which corresponds to a divisor of the form P^{-1} with some prime P satisfying $[R]=\left[\begin{pmatrix} r & s' \\ 0 & r^{-1}\end{pmatrix}\right]$. It follows from Proposition 4.1 that P must be a non Weierstrass point. Then it follows from Lemma 4.3 that there is an element λ of k satisfying $[R]=\left[\begin{pmatrix} r_\lambda & s \\ 0 & r_\lambda^{-1}\end{pmatrix}\right]$. Then it follows from Proposition 4.2 that the surjectivity holds. The fact that this map is 2:1 is easily proved using Proposition 1.4 and Lemma 4.4.

References

[1] H. Katsurada: On unramified $SL_2(F_4)$ extensions of an algebraic function field. I (to appear).

[2] ——: Ditto. II (to appear).

[3] H. Lange und U. Stuhler: Vektorbündel auf Kurven und Darstellungen der algebraischen Fundamentalgruppe. Math. Z., **156**, 73–83 (1977).