81. On Block-Schematic Steiner Systems $S(t, t+1, v)$

By Mitsuo YOSHIZAWA
College of General Education, Keio University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 12, 1980)

1. Introduction. A Steiner system $S(t, k, v)$ $(1<t<k<v)$ is called block-schematic if the blocks form an association scheme with the relations determined by size of intersection. In this note, we shall give the following theorem. The detailed proof will be given elsewhere.

Theorem. *A Steiner system $S(t, t+1, v)$ is block-schematic if and only if one of the following holds:* (i) $t=2$, (ii) $t=3$, $v=8$, (iii) $t=4$, $v=11$, (iv) $t=5$, $v=12$.

It is not difficult to check that $S(3,4,8)$, $S(4,5,11)$ and $S(5,6,12)$ are block-schematic. Moreover, $S(2,k,v)$ is also block-schematic (cf. Bose [1]). Therefore, in order to prove the theorem, it is sufficient to show that if $S(t, t+1, v)$ is block-schematic $(t\geq 3)$, then $t=3$, $v=8$, or $t=4$, $v=11$, or $t=5$, $v=12$.

2. Notation and preliminaries. For a Steiner system $S=S(t,k,v)$ we use λ_i $(0\leq i\leq t)$ to represent the number of blocks which contain the given i points of S. For a block B of S we use x_i $(0\leq i\leq k)$ to denote the number of blocks each of which has exactly i points in common with B. By a theorem of Mendelsohn [2], the number x_i depends on S, but not on the choice of a block B.

Let $B_1, \cdots, B_{\lambda_0}$ be the blocks of S. Let A_h $(0\leq h\leq k)$ be the h-adjacency matrix of S of degree λ_0 defined by

$$A_h(i,j)=\begin{cases}1 & \text{if } |B_i\cap B_j|=h,\\ 0 & \text{otherwise.}\end{cases}$$

If S is block-schematic, then $A_iA_j=\sum_{h=0}^{k}\mu(i,j,h)\,A_h(0\leq i,j\leq k)$, where $\mu(i,j,h)$ is a non-negative integer defined by the following: When there exist blocks B_p and B_q with $|B_p\cap B_q|=h$, $\mu(i,j,h)=|\{B_r|\,|B_p\cap B_r|=i,\ |B_q\cap B_r|=j,\ 1\leq r\leq\lambda_0\}|$, and when there exist no blocks B_p and B_q with $|B_p\cap B_q|=h$, $\mu(i,j,h)=0$.

3. Outline of the proof of Theorem. Let S be a block-schematic Steiner system $S(t, t+1, v)$ with $t\geq 3$. Since λ_i $(i=0,\cdots,t)$ is an integer, we have that $v-t$ is not divisible by any integer m with $1<m\leq t+1$. By a theorem of Mendelsohn [2], we have

$$x_{t-1}=\frac{(v-t-1)(t+1)t}{4},\qquad x_{t-2}=\frac{(v-t-1)(t+1)t(t-1)(v-t-5)}{36},$$

$$x_{t-3}=\frac{(v-t-1)(t+1)t(t-1)\ (t-2)\{v^2-(2t+9)v+t^2+9t+26\}}{576}.$$

Since S is block-schematic, we have

$$x_{t-1}^2=\sum_{h=0}^{t+1}\mu_h x_h, \qquad \text{where } \mu_h=\mu(t-1,t-1,h).$$

We have that $\mu_h=0$ for $h\le t-4$, $1\le\mu_{t-3}\le 12$,

$$\frac{v-t-3}{2}+4(t-1)\le\mu_{t-1}\le\frac{v-t-3}{2}+4(t-1)+\left[\frac{t-1}{2}\right],$$

$$\mu_t=0, \qquad \mu_{t+1}=x_{t-1},$$

and $9\le\mu_{t-2}\le 18$ for $t\ge 5$, and $9\le\mu_{t-2}\le 17$ for $t=3,4$ except the case $S=S(3,4,8)$, and $\mu_1=0$ for $S=S(3,4,8)$. Then, we have $3\le t\le 43$, and $36(t+1)t>(t-1)(t-2)(v-t-8)$.

Let α and β be two points of S. Let $\boldsymbol{a}_\alpha$ and $\boldsymbol{a}_\beta$ be column vectors of degree λ_0 such that

$$i\text{-th component of } \boldsymbol{a}_\alpha(\boldsymbol{a}_\beta)=\begin{cases}1 & \text{if } \alpha\in B_i\ (\beta\in B_i),\\ 0 & \text{otherwise.}\end{cases}$$

Then, A_j has an eigenvalue d_i ($j=t-1, t-2, t-3$) belonging to $\boldsymbol{a}_\alpha-\boldsymbol{a}_\beta$ such that

$$d_{t-1}=\frac{t(t-1)(v-t-1)}{4}-\frac{(t+1)t}{2},$$

$$d_{t-2}=\frac{t(t-1)(t-2)(v-t+1)(v-t-1)}{36}-\frac{(t+1)t(t-1)}{6}-(t-1)d_{t-1},$$

$$d_{t-3}=\frac{t(t-1)(t-2)(t-3)(v-t+2)(v-t+1)(v-t-1)}{576}-\frac{(t+1)t(t-1)(t-2)}{24}-\frac{(t-1)(t-2)d_{t-1}}{2}-(t-2)d_{t-2},$$

and

$$d_{t-1}^2=\mu_{t-3}d_{t-3}+\mu_{t-2}d_{t-2}+\mu_{t-1}d_{t-1}+x_{t-1}.$$

By the above informations, we get the following by computer calculations: S satisfies one of the following seven cases.

	t	v	x_{t-1}	x_{t-2}	x_{t-3}	μ_{t-1}	μ_{t-2}	μ_{t-3}	d_{t-1}	d_{t-2}	d_{t-3}
(1)	3	8	12	0	1	10	0	12	0	0	−1
(2)	3	10	18	8	3	11	9	12	3	−2	−2
(3)	3	14	30	40	20	13	9	6	9	−2	−8
(4)	4	11	30	20	15	15	15	8	8	−2	−7
(5)	4	15	50	100	100	17	11	5	20	10	−20
(6)	5	12	45	40	45	20	18	8	15	0	−15
(7)	5	16	75	200	300	22	12	5	35	40	−20

By [3], the cases (3), (5) and (7) do not hold. And it is easy to check that the case (2) does not hold.

References

[1] R. C. Bose: Strongly regular graphs, partial geometries, and partially balanced designs. Pacific J. Math., **13**, 389–419 (1963).
[2] N. S. Mendelsohn: A theorem on Steiner systems. Can. J. Math., **22**, 1010–1015 (1970).
[3] N. S. Mendelsohn and S. H. Y. Hung: On the Steiner systems $S(3,4,14)$ and $S(4,5,15)$. Utilitas Math., **1**, 5–95 (1972).