# 8. On the Cohomology of Q-Divisors

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In this paper we shall define the cohomology groups of divisors with coefficients in the field of rational numbers $\boldsymbol{Q}$ and prove the Kodaira vanishing theorem in this case. As an application we shall prove the invariance of the logarithmic plurigenera $\bar{P}_n(X)$ under deformations when $X$ is a surface of logarithmic general type. This paper is based on the idea of Miyaoka [4].

**1.** Let $X$ be a non-singular projective algebraic variety defined over the complex number field $\boldsymbol{C}$. A $\boldsymbol{Q}$*-divisor* $D$ is an element of $\mathrm{Div}(X)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$. If $D=\sum d_iD_i$, where the $d_i\in\boldsymbol{Q}$ and the $D_i$ are prime divisors on $X$, we write $[D]=\sum[d_i]D_i$, where [ ] denotes the integral part. For such a $D$ we know by Bloch and Gieseker that there is a finite Galois cover $\pi:\tilde{X}\to X$ with $\tilde{X}$ non-singular and projective such that the pull back $\pi^*D$ is integral, i.e., $\pi^*D\in\mathrm{Div}(X)$. We define $H^i(X,D)=H^i(\tilde{X},\pi^*D)^G$, where $G=\mathrm{Gal}(\tilde{X}/X)$. This is well defined because of

**Lemma 1.** *Let $f:X\to Y$ be a finite Galois cover of non-singular projective algebraic varieties and let $D\in\mathrm{Div}(Y)$. Then*

$$H^i(Y,D)\xrightarrow{\sim}H^i(X,f^*D)^G$$

*by the canonical homomorphism, where* $G=\mathrm{Gal}(X/Y)$.

**Proof.** Note that our characteristic is zero. Since the functor $A\mapsto A^G$ is exact for divisible $G$-modules, we have $H^i(X,f^*D)^G=H^i(X,(f^*D)^G)$, and the latter is isomorphic to $H^i(Y,D)$. Q.E.D.

By the same reason we get

**Lemma 2.** $H^i(X,D)=H^i(X,[D])$.

**Theorem 1.** *Let $H$ be an ample $\boldsymbol{Q}$-divisor on $X$, that is, some integral multiple $nH$ is ample on $X$. Then*

$$H^i(X,[-H])=0 \qquad \textit{for } i<\dim X.$$

**Proof.** Take $\pi:\tilde{X}\to X$ as above. By the lemmas $H^i(X,[-H])=H^i(X,-H)=H^i(\tilde{X},-\pi^*H)^G$. Since $\pi^*H$ is ample on $\tilde{X}$, we have $H^i(\tilde{X},-\pi^*H)=0$ for $i<\dim X$ by the usual Kodaira vanishing theorem. Q.E.D.

**2.** For the terminology see [1] and [2].

**Theorem 2.** *Let $X$ be a non-singular algebraic surface and let $n$ be a positive integer. Then the logarithmic pluri-genera $\bar{P}_n(X)$ are invariant under compactifiable deformations.*

**Proof.** In [2] the theorem was already proved except in case $\bar{\kappa}(X)=2$. Let $(\bar{X}_m, D_m)$ be the minimal model of $X$. By Theorem 1, $H^i(\bar{X}_m, [-n(K_m+D_m)])=0$ for $i=0$ and 1. Put $D_{[n]}=-[-nD_m]$ and $D=D_{[1]}$. Note that $D$ is a reduced divisor of normal crossing on $\bar{X}_m$. We have that $\dim H^0(\bar{X}_m, (n+1)K_m+D_{[n]})=\chi(\bar{X}_m, (n+1)K_m+D_{[n]})$, and there is an exact sequence:

$$0\to H^0(\bar{X}_m, (n+1)K_m+D_{[n]})\to H^0(\bar{X}_m, (n+1)K_m+D_{[n]}+D)$$
$$\to H^0(D, (n+1)K_m+D_{[n]}+D\,|_D)\to 0.$$

Since $nD_m\leq D_{[n]}\leq nD$ and $D-D_m$ is negative definite, the dimension of the middle term is $\bar{P}_{n+1}(X)$. On the other hand,

$$\dim H^0(D, (n+1)K_m+D_{[n]}+D\,|_D)$$
$$=\chi(D, (n+1)K_m+D_{[n]}+D\,|_D)-\dim H^0(D, -nK_m-D_{[n]}\,|_D).$$

A section of $(-nK_m-D_{[n]}\,|_D)$ has a support on a tree of rational curves, where $\dim H^0$ depends only on the degree, or on an isolated elliptic curve or on a cycle of rational curves, where $D_m$ and $D$ are the same. Thus, $\bar{P}_{n+1}(X)$ is deformation invariant. The invariance of $\bar{P}_1(X)$ follows from the theory of mixed Hodge structures (cf. [1]).

## References


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