74. ℰ-Well Posedness of Mixed Initial-Boundary Value Problem with Constant Coefficients in a Quarter Space. II

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1. Introduction. In this paper, we consider the following mixed problem:

(1.1) $A(D)u(t, x, y) = f(t, x, y), \qquad (t, x, y) \in X,$

(1.2) $B_j(D)u(t, x, y)|_{x=0} = g_j(t, x, y), \quad j = 1, \cdots, q, \quad (t, y) \in \mathbf{R}^{n+1},$

(1.3) $u(t, x, y) = 0, \quad (t, x, y) \in X, \quad t \leqslant 0,$

where $X = \{(t, x, y) \in \mathbf{R}^{n+2}; \ x > 0\}, \quad D = (D_t, D_x, D_y) = (-\sqrt{-1}\partial/\partial t, -\sqrt{-1}\partial/\partial x, -\sqrt{-1}\partial/\partial y)$ and $A(D), B_j(D), j = 1, \cdots, q$, are differential operators with constant coefficients.

Definition 1.1. The mixed problem (1.1)–(1.3) is said to be ℰ-well posed if for arbitrary data $f \in C^\infty(\overline{X})$, $g_j \in C^\infty(\mathbf{R}^{n+1})$ such that $f = 0$, $g_j = 0$ when $t \leqslant 0$, there exists one and only one solution $u \in C^\infty(\overline{X})$ of the system of the equations (1.1)–(1.3).

Remark 1.2. According to the definition of ℰ-well posedness, we have that if data vanish when $t \leqslant T(T \geqslant 0)$, then the solution also vanishes when $t \leqslant T$. From this point of view, we can consider the variable t as time.

Sakamoto [3] and the author [5] gave a necessary and sufficient condition for ℰ-well posedness of the mixed problem (1.1)–(1.3) under the assumption that $A^0(1, 0, 0) \neq 0$ (A^0 is the principal part of A). The purpose of this paper is to give a necessary and sufficient condition for ℰ-well posedness of the mixed problem (1.1)–(1.3) under the following assumption:

Assumption 1.3. $A^0(\tau, \zeta, 0) \not\equiv 0.$

Recently Nishitani [2] has studied the half space case under the same assumption as in Assumption 1.3. The author was stimulated by his work. The author would like to express his sincere gratitude to Prof. T. Nishitani, and also to Profs. M. Matsumura and S. Wakabayashi for their valuable advice.

2. Hyperbolicity of the operator A. In view of the closed graph theorem, we have that if the mixed problem (1.1)–(1.3) is ℰ-well posed, then there are positive constants T, Y, Z, C and an integer N such that

(2.1) $\displaystyle |u(1, 1, 0)| \leqslant C \Big\{ \sum_{j+k+|\alpha| \leqslant N} \sup_{0 \leqslant x \leqslant Z, |y| \leqslant Y, 0 \leqslant t \leqslant T} |D_t^j D_x^k D_y^\alpha A(D)u|$

$$+\sum_{k=1}^{q}\sum_{j+|\alpha|\leqslant N}\sup_{0\leqslant t\leqslant T,|y|\leqslant Y}|D_t^j D_y^\alpha B_k(D)u|_{x=0}|\Big\}$$

for any $u \in C^\infty(\overline{X})$ such that $u=0$ when $t\leqslant 0$.

Lemma 2.1. *If the mixed problem* (1.1)–(1.3) *is $\mathcal{E}$-well posed, then there exists $\varepsilon>0$ such that for each δ $(0<\delta<\varepsilon)$ there is P_δ with*

(2.2) $A(\tau,\zeta,\eta)\neq 0$ *for* $\mathrm{Im}\,\tau<-P_\delta\,(\log\,(1+|\tau|+|\zeta|+|\eta|)+1),$
$$\mathrm{Im}\,\zeta=\delta\,\mathrm{Im}\,\tau,\ \eta\in\boldsymbol{R}^n.$$

Proof. We take ε such that $0<\varepsilon Z<1$ and show that this lemma holds with this ε. Suppose by way of contradiction that there is $\delta>0$ $(0<\delta<\varepsilon)$ such that for every p there exist τ_p,ζ_p,η_p with $A(\tau_p,\zeta_p,\eta_p)=0$ and $\mathrm{Im}\,\tau_p<-p\,(\log\,(1+|\tau_p|+|\zeta_p|+|\eta_p|)+1)$, $\mathrm{Im}\,\zeta_p=\delta\,\mathrm{Im}\,\tau_p$, $\eta_p\in\boldsymbol{R}^n$. As we assume that the mixed problem (1.1)–(1.3) is $\mathcal{E}$-well posed, there is a solution $v_p \in C^\infty(\overline{X})$ of the following equations: $A(D)v_p=0$, $(t,x,y) \in X$; $B_j(D)v_p|_{x=0}=\psi(t)B_j(\tau_p,\zeta_p,\eta_p)\,\exp\,(\sqrt{-1}(\tau_p t+\eta_p y))$, $j=1,\cdots,q$, $(t,y) \in \boldsymbol{R}^n$; $v_p\equiv 0$, $t\leqslant 1$, where we choose $\psi(t) \in C^\infty(\boldsymbol{R}^1)$ so that $\psi(t)\equiv 0$ if $t\leqslant 1$ and $\equiv 1$ if $t\geqslant 1+\delta/2$. Choose $\phi(t) \in C^\infty(\boldsymbol{R}^1)$ so that $\phi(t)\equiv 1$ if $t\geqslant 1/2$ and $\equiv 0$ if $t\leqslant 0$. Put $u_p=[\exp\sqrt{-1}(\tau_p t+\zeta_p x+\eta_p y)]\phi(t)-v_p(t,x,y)$. If we apply (2.1) to u_p, we have that $\mathrm{Im}\,\tau_p>-C\log\,(1+|\tau_p|+|\zeta_p|+|\eta_p|)$ for some positive constant C. This leads to a contradiction, which completes the proof.

Note that it follows from Assumption 1.3 that there exists $\varepsilon'>0$ such that $A^0(1,\delta,0)\neq 0$ for any δ with $0<\delta<\varepsilon'$. By this fact, Lemma 2.1 and Seidenberg-Tarski's lemma, we have

Theorem 2.2. *Suppose that Assumption 1.3 holds. If the mixed problem* (1.1)–(1.3) *is $\mathcal{E}$-well posed, then we have:*

(A-I)   *there exists $\varepsilon>0$ such that $A(\tau,\zeta,\eta)$ is hyperbolic with respect to $(1,\delta,0)$ for any δ with $0<\delta<\varepsilon$.*

3. Main results. First we assume that Assumption 1.3 holds and that the mixed problem (1.1)–(1.3) is $\mathcal{E}$-well posed. By a change of variables:

(3.1) $x'=x,\quad t'=t+\delta^0 x,\quad y'=y,$

we have that $\tau=\tau'$, $\zeta=\delta^0\tau'+\zeta'$, $\eta=\eta'$, where (τ,ζ,η) and (τ',ζ',η') are the dual variables of (t,x,y) and (t',x',y') with respect to the usual inner product in $\boldsymbol{R}^{n+2}$, respectively, and δ^0 is a fixed positive number such that $0<\delta^0<\varepsilon$. In view of Theorem 2.2, $A'(\tau',\zeta',\eta')=A(\tau',\delta^0\tau'+\zeta',\eta')$ is hyperbolic with respect to $(1,0,0)$. Moreover, the mixed problem (1.1)–(1.3) and the following problem (3.2)–(3.4) are equivalent:

(3.2) $A'(D_{t'},D_{x'},D_{y'})u(t',x',y')=f(t',x',y')$, $x'>0$, $y' \in \boldsymbol{R}^n$,

(3.3) $B_j'(D_{t'},D_{x'},D_{y'})u(t',x',y')|_{x'=0}=g_j(t',y')$, $j=1,\cdots,q$, $y' \in \boldsymbol{R}^n$,

(3.4) $u(t',x',y')=0$, $t'\leqslant\delta^0 x'$, $x'>0$, $y' \in \boldsymbol{R}^n$,

where $B_j'(\tau',\zeta',\eta')=B_j(\tau',\delta^0\tau'+\zeta',\eta')$. Put $A'(\tau',\zeta',\eta')=\sum_{j=0}^m q_j(\tau',\eta')$ $(\zeta')^{m-j}$. In view of the localization theorem due to [1], we have that $q_0(\tau',\eta')$ is hyperbolic with respect to $(1,0) \in \boldsymbol{R}^{n+1}$. Thus, we can denote

the roots with positive imaginary part of the equation $A'(\tau', \zeta', \eta') = 0$ by $\zeta_j^+(\tau', \eta')$, $j = 1, \cdots, p$, when $-\mathrm{Im}\,\tau'$ is large enough and $\eta' \in \boldsymbol{R}^n$. Note that p is a constant number. Put $A^+(\tau', \zeta', \eta') = \prod_{j=1}^p (\zeta' - \zeta_j^+(\tau', \eta'))$. By the same method as in [5] we have:

(A-II) $p = q$.

Thus, we can define the Lopatinskiĭ determinant as follows:

$$R(\tau', \eta') = \det\left(\frac{1}{2\pi\sqrt{-1}} \oint \frac{B_j'(\tau', \zeta', \eta')(\zeta')^{k-1}}{A^+(\tau', \zeta', \eta')} d\zeta'\right)_{j,k=1,\cdots,p}.$$

By the same method as in [3], we have:

(A-III) $R(\tau', \eta')$ is hyperbolic with respect to $(1, 0) \in \boldsymbol{R}^{n+1}$.

Conversely, assume that the conditions (A-I)–(A-III) hold. According to the method due to [3], we can construct a solution $u \in C^\infty(\bar{X})$ of the equations (3.2) and (3.3) and show that there exists a closed proper cone Γ with its vertex at the origin such that $\Gamma - \{0\} \subset \{t' > \delta^0 x'\}$ and $\mathrm{supp}\,u \subset (\mathrm{supp}\,f + \Gamma) \cup \{\bigcup_{j=1}^p \mathrm{supp}\,(g_i \otimes \delta(x)) + \Gamma\}$. Moreover, we can show the uniqueness of the solution of the mixed problem (3.2)–(3.4) by the same way as in the proof of [4, Proposition 5.1]. Summing up, we have showed

Main Theorem. *Suppose that Assumption 1.3 holds. Then the mixed problem (1.1)–(1.3) is $\mathcal{E}$-well posed if and only if the conditions (A-I)–(A-III) hold.*

Remark 3.1. By (3.1) and applying the results due to Wakabayashi [6], [7], [8], we can study the propagation of singularities of fundamental solutions of the mixed problem (1.1)–(1.3).

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