73. *On the Variation of Periods of Holomorphic Γ_{h0}-Reproducing Differentials*

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1. Let R^0 be an arbitrary Riemann surfaces and fix a 1-cycle c and a Beltrami differential μ on R^0 arbitrarily. For every t with $0\leq t<1$ we denote by f^t and R^t the quasiconformal mapping from R^0 with the complex dilatation $t\mu$ and the Riemann surface $f^t(R^0)$, respectively, and denote the 1-cycle $f^t(d)$ by the same d for every 1-cycle d on R^0.

Now let θ_c^t be the holomorphic Γ_{h0}-reproducing differential for a given c on R^t. (Cf. [2, §1.5], and recall that $\theta_c^t=\theta_c(\Gamma_{h0}(R^t))$ in the notation of that paper.) Then the main purpose of this paper is to show the following

Theorem 1. *For every* 1-*cycle* d, *we have that*

$$\int_d \theta_c^t-\int_d \theta_c^0=t\cdot\mathrm{Re}\iint_{R^0}\mu\cdot\theta_c^0\cdot\theta_d^0+O(t^2).$$

Corollary. *When* $\theta_c^0\not\equiv 0$, *then it holds that*

$$\frac{d}{dt}\|\theta_c^t\|_{R^t}(0)=t\cdot\|\theta_c^0\|_{R^0}^{-1}\cdot\mathrm{Re}\iint_{R^0}\mu(\theta_c^0)^2.$$

Because $\|\theta_c^t\|_{R^t}^2=2\int_c \theta_c^t$, Corollary follows at once from Theorem 1. Here for a holomorphic quadratic differential $\phi=a(z)dz^2$ and a Beltrami differential $\mu=\mu(z)(d\bar{z}/dz)$ on R^0, we set

$$\iint_{R^0}\mu\cdot\phi=\iint_{R^0}\mu(z)\cdot a(z)\,|dz\wedge d\bar{z}|.$$

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2. First for every d and t, let σ_d^t and ω_d^t the reproducers of d in $\Gamma_{h0}(R^t)$ and $\Gamma_h(R^t)$, respectively, and set $\theta=\theta_c^t\circ f^t-\theta_c^0$, $\theta_1=\sigma_c^t\circ f^t-\sigma_c^0$ and $i\theta_2=\theta-\theta_1$. Recall that $\theta_d^t=\sigma_d^t+i{}^*\sigma_d^t$, and the following facts are known.

Lemma 1. 1) $\theta_1\in\Gamma_{c0}(R^0)$, 2) $(\theta_2,\omega_d^0)_{R^0}=0$.

Proof. 1) follows at once from [3, Theorem 3], and by [3, Theorem 4], we have $(({}^*\sigma_c^t)\circ f^t,\omega_d^0)_{R^0}=({}^*\sigma_c^t,\omega_c^t)_{R^t}=c\times d=({}^*\sigma_c^0,\omega_d^0)_{R^0}$. Q.E.D.

Lemma 2. 1) $\theta_2\in\Gamma_e(R^0)$, 2) $(\theta_1,{}^*\theta_2)=0$, 3) $(\theta_1,\omega_d^0)_{R^0}=(\theta_1,\sigma_d^0)_{R^0}$.

Proof. Because $\{\omega_d^0: d$ is any 1-cycle on $R^0\}$ spans $\Gamma_{h0}^*(R^0)$, 1) follows from Lemma 1, 2) and facts that $\theta_2\in\Gamma_c(R^0)$ and $\Gamma_c(R^0)=\Gamma_{h0}^*(R^0)+\Gamma_e(R^0)$. And because $\Gamma(R^0)=\Gamma_{c0}(R^0)+\Gamma_e^*(R^0)$, 2) follows from

Lemmas 1, 1) and 2, 1), and 3) follows from Lemma 1, 1) and the fact that $\omega_d^0-\sigma_d^0\in\Gamma_{he}^*(R^0)$. Q.E.D.

Now by Lemma 2, 2) we have $(\theta, {}^*\theta)=0$, hence by the same argument as in the proof of [2, Theorem 1] (cf. [2, Theorem 3]) we can show the following

Theorem 2. *Letting* $\|\mu\|_\infty=\mathrm{ess}\cdot\sup_{R^0}|\mu|=k\ (<1)$, *we have that*

$$\|\theta_c^t\circ f^t-\theta_c^0\|_{R^0}\le\frac{2tk}{1-tk}\|\theta_c^0\|_{R^0}.$$

3. The proof of Theorem 1. First by [3, Theorem 4] it holds that

$$I=\int_d\theta_c^t-\int_d\theta_c^0=(\theta_c^t,\omega_d^t)_{R^t}-(\theta_c^0,\omega_d^0)_{R^0}$$
$$=(\theta_c^t\circ f^t,\omega_d^0)_{R^0}-(\theta_c^0,\omega_d^0)_{R^0}=(\theta,\omega_d^0)_{R^0}.$$

Hence by Lemmas 1, 2) and 2, 3) we have that

$$I=(\theta_1,\omega_d^0)_{R^0}=(\theta_1,\sigma_d^0)_{R^0}.$$

On the other hand, $\mathrm{Re}\,(\theta,\bar\theta_d^0)=(\theta_1,\sigma_d^0)_{R^0}-(\theta_2,{}^*\sigma_d^0)_{R^0}$, hence by Lemma 2, 1) we have that

$$I=\mathrm{Re}\,(\theta,\bar\theta_d^0)=\mathrm{Re}\iint_{R^0}\theta\wedge{}^*\theta_d^0$$
$$=\mathrm{Re}\iint_{R^0}a_c^t(f^t(z))\cdot f_{\bar z}^t(z)d\bar z\wedge(-i)a_d^0(z)dz$$
$$=\mathrm{Re}\iint_{R^0}t\mu(z)\cdot a_c^t(f^t(z))\cdot f_z^t(z)\cdot a_d^0(z)\,|dz\wedge d\bar z|,$$

where, letting z^t and $z=z^0$ be the local parameter on R^t and R^0 respectively, we set $\theta_c^t=a_c^t(z^t)dz^t$ and $\theta_d^0=a_d^0(z)dz$.

Since by Theorem 2 it holds that

$$\left|\iint_{R^0}\mu(z)\cdot a_c^t(f^t(z))\cdot f_z^t(z)\cdot a_d^0(z)\,|dz\wedge d\bar z|-\iint_{R^0}\mu(z)\cdot a_c^0(z)\cdot a_d^0(z)\,|dz\wedge d\bar z|\right|$$
$$\le\|\mu\|_\infty\cdot\|\theta_d^0\|_{R^0}\cdot\|\theta_c^t\circ f^t-\theta_c^0\|_{R^0}=O(t),$$

we conclude that

$$I=t\cdot\mathrm{Re}\iint_{R^0}\mu\cdot\theta_c^0\cdot\theta_d^0+O(t^2).$$

Thus we have shown Theorem 1.

4. Remarks. Prof. Y. Kusunoki proved in [1] a similar results as Theorem 1 for periods of normal differentials on Riemann surfaces of class O'', and showed the complex differentiability of period matrix on the Teichmüller space of such a surface with respect to the Bers' coordinates.

And for the holomorphic reproducing differential $\omega_c^t+i{}^*\omega_c^t$, Prof. K. Oikawa proved the same formula as in Theorem 1 in the case that the Beltrami differential μ has a compact support, and the general case can be treated similarly as above.

References

[1] Kusunoki, Y.: Differentiability of period matrices in Teichmüller spaces (to appear).
[2] Kusunoki, Y., and Taniguchi, M.: A continuity property of holomorphic differentials under quasiconformal deformations (to appear).
[3] Minda, C. D.: Square integrable differentials on Riemann surfaces and quasiconformal mappings. Trans. Amer. Math. Soc., **195**, 365–381 (1974).