72. *Remarks on the Isomorphisms of Certain Spaces of Harmonic Differentials induced from Quasiconformal Homeomorphisms*

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Introduction. A. Marden [2] remarked that a quasiconformal homeomorphism of Riemann surfaces naturally induced an isomorphism $f^{\#}$ of the corresponding Hilbert spaces of square integrable differentials and also an isomorphism $f_h^{\#}$ of the Hilbert subspaces whose elements are harmonic differentials. Further he showed that the isomorphisms preserve several known important subspaces. D. Minda [3] investigated some other subspaces from this point of view and gave certain applications. In his article, it is remarked that $f_h^{\#}$ does not always preserve the subspaces Γ_{ae}, Γ_s and Γ_{ho}^*. He asked whether any of the classes Γ_{hse}^*, Γ_{he}^* and Γ_{hm}^* is preserved by $f_h^{\#}$ in general.

The purpose of this note is to show that $f_h^{\#}$ does not always preserve these classes.

1. Let R be a Riemann surface and $\Gamma=\Gamma(R)$ be the Hilbert space of square integrable complex differentials on R, where the inner product is given by the form:

$$(\omega_1, \omega_2)=(\omega_1, \omega_2)_R=\iint_R \omega_1\wedge\bar{\omega}_2^*$$
$$=i\iint_R (a_1\bar{a}_2+b_1\bar{b}_2)dzd\bar{z},$$

where $\omega_j=a_jdz+b_jd\bar{z}$ $(j=1,2)$ in terms of a local parameter z. As for the notations of subspaces of Γ we shall follow Ahlfors-Sario [1] and make use of basic results in this reference.

2. Now suppose that $f:R'\to R$ is a quasiconformal mapping of a Riemann surface R' onto a Riemann surface R. Then f induces an isomorphism $f^{\#}:\Gamma(R)\to\Gamma(R')$ so that

$$f^{\#}(\omega)=[A(f)f_\zeta+B(f)(\bar{f})_\zeta]d\zeta+[A(f)f_{\bar{\zeta}}+B(f)(\bar{f})_{\bar{\zeta}}]d\bar{\zeta}$$

in a neighbourhood of p', where $\omega=A(z)dz+B(z)d\bar{z}$ in terms of a local parameter z in a neighbourhood of $p=f(p')$, ζ a local parameter about p' and $f_\zeta, f_{\bar{\zeta}}, (\bar{f})_\zeta, (\bar{f})_{\bar{\zeta}}$ are distributional derivatives of f and $\bar{f}$. Let P_h denote the projection from Γ to Γ_h whose elements are harmonic differentials. Then the mapping $f_h^{\#}=P_h\circ f^{\#}$ gives an isomorphism from $\Gamma_h(R)$ to $\Gamma_h(R')$ (cf. [2], [3]). Let $\sigma(C')^*\in\Gamma_{ho}(R')^*$ be the period reproducing differential for a cycle C' on R' and $\sigma(f(C'))^*\in\Gamma_{ho}(R)^*$ be the one for the cycle $f(C')$ on R. We know the following propositions due to A. Marden and D. Minda (cf. [2], [3]).

Proposition 1. *If* $f: R' \to R$ *and* $g: R \to R''$ *are quasiconformal mappings, then* $(g \circ f)^{\#} = f^{\#} \circ g^{\#}$ *and* $(g \circ f)_h^{\#} = f_h^{\#} \circ g_h^{\#}$. *If* f *is an identity mapping, then* $f^{\#}$ *and* $f_h^{\#}$ *are identity mappings of* Γ *and* Γ_h *respectively.*

Proposition 2. *Let* f *be a quasiconformal mapping form* R' *to* R. *Then*

(i) $(\tau, \sigma(f(C'))^*)_R = (f^{\#}(\tau), \sigma(C')^*)_{R'}$ *for* $\tau \in \Gamma_c(R)$,
$(\omega, \sigma(f(C'))^*)_R = (f_h^{\#}(\omega), \sigma(C')^*)_{R'}$ *for* $\omega \in \Gamma_h(R)$,

(ii) $(f^{\#}(\Gamma_x(R)) = \Gamma_x(R')$, *where* $\Gamma_x = \Gamma_c, \Gamma_{se}, \Gamma_e, \Gamma_{eo}$,
$(f_h^{\#}(\Gamma_y(R)) = \Gamma_y(R')$, *where* $\Gamma_y = \Gamma_{hse}, \Gamma_{he}, \Gamma_{ho}, \Gamma_{hm}$.

3. We first remark the following

Lemma 3. *Let* f *be a quasiconformal mapping from* R' *to* R. *Then*

$$(f^{\#}(\tau_1)^*, f^{\#}(\tau_2^*))_{R'} = (\tau_1, \tau_2)_R \qquad \text{for } \tau_1, \tau_2 \in \Gamma(R),$$
$$(f_h^{\#}(\omega_1)^*, f_h^{\#}(\omega_2^*))_{R'} = (\omega_1, \omega_2)_R \qquad \text{for } \omega_1, \omega_2 \in \Gamma_h(R).$$

Proof. Let $\tau_j = A_j dz + B_j d\bar{z}$ $(j=1,2)$. We have

$$\begin{aligned}(f^{\#}(\tau_1), f^{\#}(\tau_2^*)^*)_{R'} &= -i\iint_{R'} (A_1\bar{A}_2 + B_1\bar{B}_2)(|f_\zeta|^2 - |f_{\bar{\zeta}}|^2) d\zeta d\bar{\zeta} \\ &= -i\iint_R (A_1\bar{A}_2 + B_1\bar{B}_2) dz d\bar{z} \\ &= -(\tau_1, \tau_2)_R.\end{aligned}$$

Thus the first equality follows. Next by the orthogonal decomposition $\Gamma = \Gamma_h + \Gamma_{eo} + \Gamma_{eo}^*$, we can get the second equality.

Remark. By this lemma,

$$\begin{aligned}(\omega, \sigma(f(C'))^*)_R &= (f_h^{\#}(\omega)^*, f_h^{\#}(-\sigma(f(C'))))_{R'} \\ &= (f_h^{\#}(\omega), f_h^{\#}(\sigma(f(C')))^*)_{R'}.\end{aligned}$$

Hence we know $f_h^{\#}(\sigma(f(C')))^* = \sigma(C')^* = f^*(\sigma(f(C'))^*)$ which gives a relation between $f_h^{\#}$ and f^* induced from the ho-mapping f [3].

4. We shall prove

Proposition 4. *Let* Γ_1 *and* Γ_2 *be subspaces of* Γ_h *and* Γ_1 *be orthogonal to* Γ_2. *If* $f_h^{\#}(\Gamma_1(R)^* + \Gamma_2(R)^*) = \Gamma_1(R')^* + \Gamma_2(R')^*$ *and* $f_h^{\#}(\Gamma_1(R)) = \Gamma_1(R')$, *then* $f_h^{\#}(\Gamma_2(R)^*) = \Gamma_2(R')^*$.

Proof. If $\omega_2 \in \Gamma_2(R)$, then $f_h^{\#}(\omega_2^*) \in \Gamma_1(R')^* + \Gamma_2(R')^*$. By Lemma 3, for any $\omega_1 \in \Gamma_1(R)$

$$(f_h^{\#}(\omega_1)^*, f_h^{\#}(\omega_2^*))_{R'} = (\omega_1, \omega_2)_R = 0.$$

Hence $f_h^{\#}(\omega_2^*)$ is orthogonal to $f_h^{\#}(\Gamma_1(R))^* = \Gamma_1(R')^*$ and $f_h^{\#}(\Gamma_2(R)^*) \subset \Gamma_2(R')^*$. With the aid of Proposition 1, we can apply this to $(f^{-1})_h^{\#}$ and we have

$$(f^{-1})_h^{\#}(\Gamma_2(R')^*) \subset \Gamma_2(R)^*,$$
$$\Gamma_2(R')^* = f_h^{\#} \circ (f^{-1})_h^{\#}(\Gamma_2(R')^*) \subset f_h^{\#}(\Gamma_2(R)^*).$$

Thus we get the conclusion.

If we make use of the orthogonal decompositions:

$$\Gamma_h=\Gamma_{he}\cap\Gamma_{he}^*+\Gamma_s=\Gamma_{he}+\Gamma_{ho}^*=\Gamma_{hse}+\Gamma_{hm}^*,$$

we have

Corollary 5. (i) $f_h^\#(\Gamma_s(R))=\Gamma_s(R')\Longleftrightarrow f_h^\#(\Gamma_{he}(R)\cap\Gamma_{he}(R)^*)$
$=\Gamma_{he}(R')\cap\Gamma_{hc}(R')^*$,

(ii) $f_h^\#(\Gamma_{ho}(R)^*)=\Gamma_{ho}(R')^*\Longleftrightarrow f_h^\#(\Gamma_{he}(R)^*)=\Gamma_{he}(R')^*$,

(iii) $f_h^\#(\Gamma_{hm}(R)^*)=\Gamma_{hm}(R')^*\Longleftrightarrow f_h^\#(\Gamma_{hse}(R)^*)=\Gamma_{hse}(R')^*$.

5. Now there exist Riemann surfaces $R\notin O_{AD}$ and $R'\in O_{AD}$ which have a quasiconformal mapping f from R' to R. Then we clearly have $f_h^\#(\Gamma_{ae}(R))\neq\Gamma_{ae}(R')$ and $f_h^\#(\Gamma_{he}(R)\cap\Gamma_{he}(R)^*)\neq\Gamma_{he}(R')\cap\Gamma_{he}(R')^*$. So by Corollary 5, $f_h^\#(\Gamma_s(R))\neq\Gamma_s(R')$. Since $\Gamma_s=\mathrm{Cl}\,(\Gamma_{ho}+\Gamma_{ho}^*)$ and $f_h^\#(\Gamma_{ho}(R))=\Gamma_{ho}(R')$, we have $f_h^\#(\Gamma_{ho}(R)^*)\neq\Gamma_{ho}(R')^*$ (cf. [3]). It follows by Corollary 5 that $f_h^\#(\Gamma_{he}(R)^*)\neq\Gamma_{he}(R')^*$.

Next we give an example that $f_h^\#$ does not preserve Γ_{hm}^* and Γ_{hse}^*. Take rectangles

$$R_i=\{(x,y)\,;\,-a_i\le x\le a_i,\ -b_i\le y\le b_i\}\qquad(i=1,2)$$

and discs D_i and D_i' in R_i;

$$D_1=\{(x,y)\,;\,x^2+(y-d_1)^2\le r_1^2\},$$
$$D_1'=\{(x,y)\,;\,x^2+(y+d_1)^2\le r_1^2\},$$
$$D_2=\{(x,y)\,;\,(x-d_2)^2+y^2\le r_2^2\},$$
$$D_2'=\{(x,y)\,;\,(x+d_2)^2+y^2\le r_2^2\}.$$

Denote by A_i, A_i' the vertical sides of R_i and by B_i, B_i' the horizontal sides. We identify A_i and B_i with A_i' and B_i' respectively, and get a torus from R_i. Further we remove the discs D_i and D_i' from the torus and denote it by T_i. Let w_i be a harmonic function on T_i such that $w_i(x,y)=1$ on ∂D_i, $=0$ on $\partial D_i'$. Then $\Gamma_{hm}(T_i)=\{cdw_i\}$. From the symmetricity, we have

$$\int_{A_1}dw_1^*=0\quad\text{and}\quad\int_{A_2}dw_2^*\neq 0.$$

On the other hand, there exists a quasiconformal mapping f from T_1 to T_2 so that $f(\partial D_1)=\partial D_2$, $f(\partial D_1')=\partial D_2'$, $f(A_1)=A_2$ and $f(B_1)=B_2$. By Proposition 2, $\int_{A_2}f_h^\#(dw_1^*)=0$. Hence we have $f_h^\#(\Gamma_{hm}(T_1)^*)\neq\Gamma_{hm}(T_2)^*$ and $f_h^\#(\Gamma_{hse}(T_1)^*)\neq\Gamma_{hse}(T_2)^*$. Further by $\Gamma_{hse}=\Gamma_{hse}\cap\Gamma_{hse}^*+\Gamma_{hm}$, we have $f_h^\#(\Gamma_{hse}(T_1)\cap\Gamma_{hse}(T_1)^*)\neq\Gamma_{hse}(T_2)\cap\Gamma_{hse}(T_2)^*$. Thus we have

Proposition 6. *The classes Γ_{hm}^*, Γ_{hse}^*, $\Gamma_{hse}\cap\Gamma_{hse}^*$ and Γ_{he}^* are not always preserved by $f_h^\#$.*

6. Finally we remark that

Proposition 7. *Let f be an extremal quasiconformal homeomorphism from a compact Riemann surface R' to R. Assume that $f_h^\#(\sigma(f(C'))^*)=\sigma(C')^*$ for any cycle C' in R'. Then f is a conformal mapping.*

Proof. We have $f_h^\#(\sigma(f(C'))+i\sigma(f(C'))^*)=\sigma(C')+i\sigma(C')^*$. Hence the normalized holomorphic differentials on R is mapped by $f_h^\#$ to those on R'. They are the same periods for corresponding cycle, i.e., their period matrices coinside. Thus by the Torelli's theorem, R' is conformal to R by f.

References

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