70. *Deformation of Linear Ordinary Differential Equations. IV*

By Tetsuji MIWA
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1980)

In this note exploiting quantum field operators we construct an isomonodromy family with a prescribed monodromy data. This approach was initiated by Sato, Miwa and Jimbo [1] in the case of regular singularities. As for irregular singularities some special cases have been treated in [2], [3]. Here we consider the following general case; we construct an $m\times m$ matrix $Y(x_0, x)$ normalized as $Y(x_0, x_0)=1$ which enjoys the monodromy property with respect to x with the following monodromy data [4], [5]

$$(1)\qquad \begin{array}{l} a_1\,;\, T^{(1)}_{-r_1}, \cdots, T^{(1)}_0, S^{(1)}_1, \cdots, S^{(1)}_{2r_\nu}, C^{(1)}, \\ \qquad\vdots \\ a_n\,;\, T^{(n)}_{-r_n}, \cdots, T^{(n)}_0, S^{(n)}_1, \cdots, S^{(n)}_{2r_\nu}, C^{(n)}. \end{array}$$

Here $a_1, \cdots, a_n$ are distinct points in $\boldsymbol{C}$. r_ν is the rank of the irregular singularity at a_ν. $T^{(\nu)}_{-r_\nu}, \cdots, T^{(\nu)}_0$ are the exponent matrices at a_ν. We assume that if $r_\nu\geqq 1$,

$$(2)\qquad t^{(\nu)}_{-r_\nu,\beta}\neq t^{(\nu)}_{-r_\nu,\beta} \qquad \text{for } \alpha\neq\beta,$$

where $T^{(\nu)}_{-r_\nu}=(t^{(\nu)}_{-r_\nu\alpha}\delta_{\alpha\beta})_{\alpha,\beta=1,\cdots,m}$. $S^{(\nu)}_1, \cdots, S^{(\nu)}_{2r_\nu}$ are the Stokes multipliers with respect to the sectors $S^{(\nu)}_{l,\delta}$ at a_ν (see (2.38) and (2.43) in [4]). $C^{(\nu)}$ is the connection matrix from a_ν to x_0. Note that $x=\infty$ is chosen to be a regular point for $Y(x_0, x)$. We should assume the following consistency conditions.

$$(3)\qquad \sum_{\nu=1}^{n}\sum_{\alpha=1}^{m} t^{(\nu)}_{0\alpha}=0,$$

$$(4)\qquad (C^{(n)-1}e^{2\pi i T^{(1)}_0}S^{(n)-1}_{2r_n}\cdots S^{(n)-1}_1 C^{(n)}) \times\cdots\times (C^{(1)-1}e^{2\pi i T^{(1)}_0}S^{(1)-1}_{2r_1}\cdots S^{(1)-1}_1 C^{(1)})=1.$$

Under the above assumptions, we shall give a Neumann series for $Y(x_0, x)$ in (22), which is convergent if $T^{(\nu)}_{-j}$ and $S^{(\nu)}_l-1$ $(\nu=1, \cdots, n;\ j=0, 1, \cdots, r_\nu;\ l=1, \cdots, 2r_\nu)$ are sufficiently small.

We also give expressions for the characteristic matrices $G^{(\nu,\mu)(l,k)}$ $(\nu, \mu=1, \cdots, n;\ l, k\geqq 1)$ (see [6]) of the isomonodromy family. Since the characteristic matrices give rise to solutions to the non-linear deformation equations for the isomonodromy family, we thus obtain analytic expressions for these solutions. We refer the reader to [7]–[12] as for previous results on analytic expressions for solutions to Painlevé equations and their generalizations.

We exploit free fermions denoted by $\psi_\alpha(x)$, $\psi^*_\alpha(x)$, $\psi^{(\nu)}_\alpha(x)$ and $\psi^{*(\nu)}_\alpha(x)$ ($x \in \boldsymbol{R}$; $\alpha=1,\cdots,m$; $\nu=1,\cdots,n$). We define the expectation value between $\psi_\alpha(x)(\psi^*_\alpha(x))$ and one of the free fermions to be zero except for the following.

$$(5)\qquad \langle\psi^*_\alpha(x)\psi_\beta(x')\rangle=\langle\psi_\alpha(x)\psi^*_\beta(x')\rangle=\delta_{\alpha\beta}\frac{1}{2\pi}\frac{i}{x-x'+i0},$$

$$(6)\qquad \langle\psi^*_\alpha(x)\psi^{(\nu)}_\beta(x')\rangle=\langle\psi^{(\nu)}_\beta(x)\psi^*_\alpha(x')\rangle=(C^{(\nu)-1})_{\alpha\beta}\frac{1}{2\pi}\frac{-i}{x-x'-i0},$$

$$(7)\qquad \langle\psi^{*(\nu)}_\alpha(x)\psi_\beta(x')\rangle=\langle\psi_\beta(x)\psi^{*(\nu)}_\alpha(x')\rangle=C^{(\nu)}_{\alpha\beta}\frac{1}{2\pi}\frac{i}{x-x'+i0}.$$

The table of the expectation values for other pairs is given in (16) and (17). Here we need only the following.

$$(8)\qquad \begin{aligned}\langle\psi^{*(\nu)}_\alpha(x)\psi^{(\nu)}_\alpha(x')\rangle&=\langle\psi^{(\nu)}_\alpha(x)\psi^{*(\nu)}_\alpha(x')\rangle=0,\\ \langle\psi^{*(\nu)}_\alpha(x)\psi^{*(\nu)}_\alpha(x')\rangle&=\langle\psi^{(\nu)}_\alpha(x)\psi^{(\nu)}_\alpha(x')\rangle=0.\end{aligned}$$

We set

$$(9)\qquad \varphi^{(\nu)}_\alpha=e^{\rho^{(\nu)}_\alpha},\qquad \rho^{(\nu)}_\alpha=\iint dxdx'R^{(\nu)}_\alpha(x,x')\psi^{(\nu)}_\alpha(x)\psi^{*(\nu)}_\alpha(x'),$$

$$(10)\qquad R^{(\nu)}_\alpha(x,x')=\frac{1}{2\pi}\frac{i}{x-x'+i0}\frac{e^{(\nu)}_\alpha(x')}{e^{(\nu)}_\alpha(x)},$$

$$(11)\qquad e^{(\nu)}_\alpha(x)=\exp\left(\sum_{j=1}^{r_\nu}t^{(\nu)}_{-j,\alpha}\frac{(x-a_\nu)^{-j}}{(-j)}+t^{(\nu)}_{0,\alpha}\log(x-a_\nu)\right).$$

$$(12)\qquad \varphi^{(\nu,k)}_\alpha=\psi^{(\nu,k)}_\alpha e^{\rho^{(\nu)}_\alpha},\quad \psi^{(\nu,k)}_\alpha=\frac{-1}{\sqrt{2\pi i}}\int dxe^{(\nu,k)}_\alpha(x)^{-1}\psi^{(\nu)}_\alpha(x)\quad(k\geqq 1),$$

$$(13)\qquad \varphi^{(\nu,-l)}_\alpha=\psi^{*(\nu,-l)}_\alpha e^{\rho^{(\nu)}_\alpha},\quad \psi^{*(\nu,-l)}_\alpha=\frac{1}{\sqrt{2\pi i}}\int dxe^{(\nu,-l)}_\alpha(x)\psi^{*(\nu)}_\alpha(x)\quad(l\geqq 1),$$

$$(14)\qquad e^{(\nu,j)}_\alpha(x)=(x-a_\nu)^je^{(\nu)}_\alpha(x)\qquad(j\in \boldsymbol{Z}).$$

$$(15)\qquad \varphi^{(\nu,-l,k)}_\alpha=\psi^{*(\nu,-l)}_\alpha\psi^{(\nu,k)}_\alpha e^{\rho^{(\nu)}_\alpha}\qquad(l,k\geqq 1).$$

We define kernels $K^{(\nu,\mu)}_{\alpha\beta}(x,x')$ ($\nu,\mu=1,\cdots,n$; $\alpha,\beta=1,\cdots,m$) and the remaining expectation values as follows.

$$(16)\qquad \begin{aligned}K^{(\nu,\mu)}_{\alpha\beta}(x,x')&=-(C^{(\nu)}C^{(\mu)-1})_{\alpha\beta}\frac{1}{2\pi}\frac{i}{x-x'+i0}\qquad(\nu\neq\mu),\\ &=\begin{cases}\langle\psi^{*(\nu)}_\alpha(x)\psi^{(\mu)}_\beta(x')\rangle & (\nu<\mu)\\ -\langle\psi^{(\mu)}_\beta(x')\psi^{*(\nu)}_\alpha(x)\rangle & (\nu>\mu).\end{cases}\end{aligned}$$

$$(17)\qquad \begin{aligned}K^{(\nu,\nu)}_{\alpha\beta}(x,x')&=\lambda^{(\nu)}_{\alpha\beta}\delta(x-x')\qquad(\alpha\neq\beta),\\ &=\begin{cases}\langle\psi^{*(\nu)}_\alpha(x)\psi^{(\nu)}_\beta(x')\rangle & (\alpha<\beta)\\ -\langle\psi^{(\nu)}_\beta(x')\psi^{*(\nu)}_\alpha(x)\rangle & (\alpha>\beta).\end{cases}\end{aligned}$$

Here $\lambda^{(\nu)}_{\alpha\beta}$ is a complex parameter specified below. The kernel $K^{(\nu,\nu)}_{\alpha\alpha}(x,x')$ and the rest of the expectation values between $\psi^{(\nu)}_\alpha(x)$ and $\psi^{*(\nu)}_\alpha(x)$ are zero.

Now let us consider the product

$$(18)\qquad \varphi^{(1)}_1\cdots\varphi^{(1)}_m\cdots\varphi^{(n)}_1\cdots\varphi^{(n)}_m=\langle\varphi^{(1)}_1\cdots\varphi^{(1)}_m\cdots\varphi^{(n)}_1\cdots\varphi^{(n)}_m\rangle:e^\rho:,$$

$$(19)\qquad \rho=\sum_{\nu,\mu=1}^{n}\sum_{\alpha,\beta=1}^{m}\iint dxdx'R^{(\nu,\mu)}_{\alpha\beta}(x,x')\psi^{(\nu)}_\alpha(x)\psi^{*(\mu)}_\beta(x').$$

The kernel is given, at least formally, by the following Neumann series.

$$(20)\quad R^{(\nu,\mu)}_{\alpha\beta}(x,x')=\sum_{j=0}^{\infty}\int dx_1\cdots\int dx_{2j}\sum_{\nu_1,\cdots,\nu_{j-1}=1}^{n}\sum_{\alpha_1,\cdots,\alpha_{j-1}=1}^{m} \times R^{(\nu_0)}_{\alpha_0}(x_0,x_1)K^{(\nu_0,\nu_1)}_{\alpha_0\alpha_1}(x_1,x_2)R^{(\nu_1)}_{\alpha_1}(x_2,x_3)K^{(\nu_1,\nu_2)}_{\alpha_1\alpha_2}(x_3,x_4)\cdots R^{(\nu_j)}_{\alpha_j}(x_{2j},x_{2j+1}),$$

where $(\nu_0,\alpha_0)=(\nu,\alpha)$, $(\nu_j,\alpha_j)=(\mu,\beta)$, $x_0=x$ and $x_{2j+1}=x'$. Since the free fermions are defined on the real axis the integrations appearing in (20) should be on the real axis. Nevertheless in order to obtain an isomonodromy family we introduce the following modification for the contours of integration.

We assume that $\mathrm{Im}\,a_\kappa$ $(\kappa=1,\cdots,n)$ are distinct. If $\nu_{k-1}\neq\nu_k$, the contour for x_{2k-1} (resp. x_{2k}) should be $I^{(\nu_{k-1})}$ (resp. $I^{(\nu_k)}$) of Fig. 1.

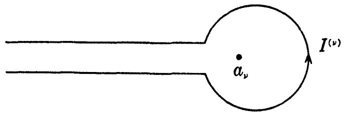

Fig. 1

If $\nu_{k-1}=\nu_k$ $(=\kappa)$, $K^{(\kappa,\kappa)}_{\gamma\gamma'}(x_{2k-1},x_{2k})$ $(\gamma=\alpha_{k-1},\gamma'=\alpha_k)$ contains $\delta(x_{2k-1}-x_{2k})$. Hence we can integrate over x_{2k-1}. Then the integrand for x_{2k} contains the factor $e^{(\kappa)}_{\gamma}(x_{2k})/e^{(\kappa)}_{\gamma'}(x_{2k})$. In the case of $r_\nu\geq 1$, by the assumption (2) there exist r_κ sectors $\mathcal{S}^{(\kappa)}_{\gamma\gamma',1},\cdots,\mathcal{S}^{(\kappa)}_{\gamma\gamma',r_\kappa}$ at a_κ along which this factor is decreasing. We choose $\mathcal{S}^{(\kappa)}_{\gamma\gamma',1},\cdots,\mathcal{S}^{(\kappa)}_{\gamma\gamma',r_\kappa}$ successively anticlockwise so that $\mathcal{S}^{(\kappa)}$ is contained in either $\mathcal{S}^{(\kappa)}_{1,\delta}\cup\mathcal{S}^{(\kappa)}_{2,\delta}$ or $\mathcal{S}^{(\kappa)}_{2,\delta}\cup\mathcal{S}^{(\kappa)}_{3,\delta}$ (see (2.38) in [4]). Then we take a contour $I^{(\kappa)}_{\gamma\gamma',l}$ from ∞ to a_κ through $\mathcal{S}^{(\kappa)}_{\gamma\gamma',l}$. $I^{(\kappa)}_{\gamma\gamma',l}$ $(l=1,\cdots,r_\kappa)$ should be chosen so that $\mathrm{Im}\,x_{2k,l_1}>\mathrm{Im}\,x_{2k,l_2}$ for $x_{2k,l_j}\in I^{(\kappa)}_{\gamma\gamma',l_j}$ $(j=1,2)(l_1<l_2)$ when $x_{2k,l_j}\to\infty_m$ (Fig. 2).

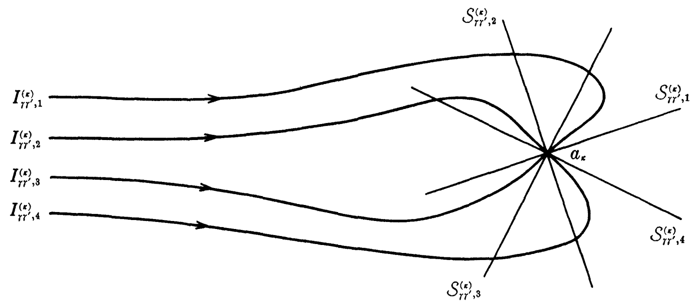

Fig. 2

Moreover we choose $\lambda^{(\kappa)}_{\gamma\gamma'}$ of (17) differently for each contour; we take $\lambda^{(\kappa)}_{l,\gamma\gamma'}$ for $I^{(\kappa)}_{\gamma\gamma',l}$ $(l=1,\cdots,r_\kappa)$. When the contours for x_{2k} and x_{2k+1} coincide with each other, we choose the contour for x_{2k+1} in the right of the contour for x_{2k}. Likewise x_0 (resp. x_{2j+1}) is supposed to be in the left of the contour for x_1 (resp. in the right of the contour for x_{2j}). With the above prescriptions for integration contours, the Neumann series

(20) is convergent for sufficiently small $\lambda_{l,\alpha\beta}^{(\nu)}$ ($\nu=1,\cdots,n$; $l=1,\cdots,r_\nu$; $\alpha,\beta=1,\cdots,m$) and $t_{-j,\alpha}^{(\nu)}$ ($\nu=1,\cdots,n$; $j=0,1,\cdots,r_\nu$; $\alpha=1,\cdots,m$).

Now let us consider the following expectation value.

(21) $$Y(x_0\,;x)_{\alpha\beta}=2\pi i(x-x_0)\langle\psi_\alpha^*(x_0):e^\rho:\psi_\beta(x)\rangle.$$

From (5)–(7) and (19) we have

(22) $$Y(x_0\,;x)_{\alpha\beta}=1+\sum_{\nu,\mu=1}^{n}\sum_{\alpha_1,\alpha_2=1}^{m}\int_{I^{(\nu)}}dx_1\int_{I^{(\mu)}}dx_2 \times(C^{(\nu)-1})_{\alpha\alpha_1}\frac{1}{2\pi}\frac{-i}{x_0-x_1-i0}R_{\alpha_1\alpha_2}^{(\nu,\mu)}(x_1,x_2)\frac{1}{2\pi}\frac{i}{x_2-x+i0}C_{\alpha_2\beta}^{(\mu)}.$$

Here x_0 (resp. x) are supposed to be outside of the contour $I^{(\nu)}$ (resp. $I^{(\mu)}$). The $m\times m$ matrix $Y(x_0\,;x)=(Y(x_0\,;x)_{\alpha\beta})_{\alpha,\beta=1,\cdots,m}$ gives the isomonodromy family normalized at x_0, i.e. $Y(x_0,x_0)=1$. The connection matrix from a_ν to x_0 is given by $C^{(\nu)}$, the Stokes multipliers $S_l^{(\nu)}$ ($l=1,\cdots,2r_\nu$) are given by

(23) $$S_l^{(\nu)}=(1-\Lambda_l^{(\nu)})^{-1},$$

where $(\Lambda_l^{(\nu)})_{\alpha\beta}=\lambda_{l,\alpha\beta}^{(\nu)}$ if $e_\alpha^{(\nu)}(x)/e_\beta^{(\nu)}(x)$ is decreasing in the sector $\mathcal{S}_{l,\delta}\cap\mathcal{S}_{l+1,\delta}$ and $(\Lambda_l^{(\nu)})_{\alpha\beta}=0$ otherwise.

The characteristic matrix $G^{(\nu,\mu)(l,k)}$ [6] is expressed as follows.

(24) $$G_{\alpha\beta}^{(\nu,\mu)(l,k)}=\begin{cases}-\langle\varphi_1^{(1)}\cdots\varphi_\beta^{(\mu,k)}\cdots\varphi_\alpha^{(\nu,-l)}\cdots\varphi_m^{(n)}\rangle/\langle\varphi_1^{(1)}\cdots\varphi_m^{(n)}\rangle & \text{if } \mu<\nu \text{ or } \nu=\mu,\ \beta<\alpha\\ \langle\varphi_1^{(1)}\cdots\varphi_\alpha^{(\nu,-l,k)}\cdots\varphi_m^{(n)}\rangle/\langle\varphi_1^{(1)}\cdots\varphi_m^{(n)}\rangle & \text{if } \nu=\mu,\ \alpha=\beta\\ \langle\varphi_1^{(1)}\cdots\varphi_\alpha^{(\nu,-l)}\cdots\varphi_\beta^{(\mu,k)}\cdots\varphi_m^{(n)}\rangle/\langle\varphi_1^{(1)}\cdots\varphi_m^{(n)}\rangle & \text{if } \nu<\mu \text{ or } \nu=\mu,\ \alpha<\beta.\end{cases}$$

Now let us consider the Schlesinger transform for $Y(x_0,x)$ of the type $\begin{Bmatrix}a_1\cdots a_n\\ L^{(1)}\cdots L^{(n)}\end{Bmatrix}$ where $L^{(\nu)}=(l_\alpha^\nu\delta_{\alpha\beta})_{\alpha,\beta=1,\cdots,m}$ ($\nu=1,\cdots,n$) such that

$$\sum_{\nu=1}^{n}\sum_{\alpha=1}^{m}l_\alpha^\nu=0$$

(see II [6]). We set

(25) $$\begin{aligned}&\bar\varphi_\alpha^{(\nu,-l)}=\psi_\alpha^{*(\nu,-1)}\cdots\psi_\alpha^{*(\nu,-l)}e^{\rho_\alpha^{(\nu)}} && (l\geqq 1),\\ &\bar\varphi_\alpha^{(\nu,0)}=\varphi_\alpha^{(\nu)},\\ &\bar\varphi_\alpha^{(\nu,k)}=\psi_\alpha^{(\nu,1)}\cdots\psi_\alpha^{(\nu,k)}e^{\rho_\alpha^{(\nu)}} && (k\geqq 1).\end{aligned}$$

Then we have (see II [6])

(26) $$\det W\begin{Bmatrix}a_1\cdots a_n\\ L^{(1)}\cdots L^{(n)}\end{Bmatrix};Y(x)\Bigr\}=\pm\frac{\langle\bar\varphi_1^{(1,l_1^1)}\cdots\bar\varphi_m^{(1,l_m^1)}\cdots\bar\varphi_1^{(n,l_1^n)}\cdots\bar\varphi_m^{(n,l_m^n)}\rangle}{\langle\varphi_1^{(1)}\cdots\varphi_m^{(n)}\rangle}$$

(27) $$Y(x_0,x)'_{\alpha\beta}=2\pi i(x-x_0)\frac{\langle\psi_\alpha^*(x_0)\bar\varphi_1^{(1,l_1^1)}\cdots\bar\varphi_m^{(n,l_m^n)}\psi_\beta(x)\rangle}{\langle\bar\varphi_1^{(1,l_1^1)}\cdots\bar\varphi_m^{(n,l_m^n)}\rangle}.$$

The logarithmic derivative of the correlation function $\langle\varphi_1^{(1)}\cdots\varphi_m^{(n)}\rangle$ is given by

(28) $$d\log\langle\varphi_1^{(1)}\cdots\varphi_m^{(n)}\rangle=-\sum_{l=1}^{\infty}\sum_{\nu_1,\cdots,\nu_l=1}^{n}\sum_{\alpha_1,\cdots,\alpha_l=1}^{m}\int\cdots\int dx_1\cdots dx_{2l} \times dR_{\alpha_1}^{(\nu_1)}(x_1,x_2)K_{\alpha_1\alpha_2}^{(\nu_1,\nu_2)}(x_2,x_3)R_{\alpha_2}^{(\nu_2)}(x_3,x_4)K_{\alpha_2\alpha_3}^{(\nu_2,\nu_3)}(x_4,x_5)\cdots R_{\alpha_l}^{(\nu_l)}(x_{2l-1},x_{2l})K_{\alpha_l\alpha_1}^{(\nu_l,\nu_1)}(x_{2l},x_1),$$

where the contours for integration should be specified as in (20). We can prove the following identity (see (33) [5]).

(29) $$\omega = d \log \langle \varphi_1^{(1)} \cdots \varphi_m^{(n)} \rangle.$$

Thus the correlation function $\langle \varphi_1^{(1)} \cdots \varphi_m^{(n)} \rangle$ coincides with the τ function.

It is a pleasure to thank Dr. M. Jimbo, Prof. M. Sato and Dr. K. Ueno for useful discussions.

References

[1] M. Sato, T. Miwa, and M. Jimbo: Publ. RIMS, Kyoto Univ., **15**, 201 (1979).
[2] ——: Proc. Japan Acad., **55**A, 267 (1979).
[3] M. Jimbo, T. Miwa, Y. Môri, and M. Sato: Physica, **1**D, 80 (1980).
[4] M. Jimbo, T. Miwa, and K. Ueno: Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. RIMS preprint, no. 319, Kyoto Univ. (1980) (to appear in Physica D.).
[5] M. Jimbo and T. Miwa: Deformation of linear ordinary differential equations. I. Ibid., no. 315, Kyoto Univ. (1980).
[6] ——: Deformation of linear ordinary differential equations. II. Ibid., no. 316, Kyoto Univ. (1980); III. Ibid., no. 329, Kyoto Univ. (1980).
[7] T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch: Phys. Rev., B, **13**, 316 (1976).
[8] B. M. McCoy, C. A. Tracy, and T. T. Wu: J. Math. Phys., **18**, 1058 (1977).
[9] M. J. Ablowitz and H. Segur: Studies in Appl. Math., **57**, 13 (1977).
[10] B. M. McCoy, C. A. Tracy, and T. T. Wu: Phys. Lett., **61**A, 283 (1977).
[11] M. Sato, T. Miwa, and M. Jimbo: Publ. RIMS, Kyoto Univ., **15**, 871 (1979).
[12] H. Flaschka and A. C. Newell: Monodromy and spectrum preserving deformation. I. Clarkson College of Tech. (1979) (preprint).