69. A Note on the Tate Conjecture for K3 Surfaces

By Takayuki ODA
Hokkaido University

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This note discusses the openness of the image of the Galois group in the second ℓ-adic cohomology of a $K3$ surface with large Picard number defined over an algebraic number field. Especially, we prove the Tate conjecture for a $K3$ surface, whose Picard number is 20 or 19.

Let X be a smooth projective geometrically irreducible surface defined over an algebraic number field k, which satisfies the conditions:

$$\Omega^2_{X/k}=\mathcal{O}_X \quad \text{and} \quad H^1(X, \mathcal{O}_X)=0.$$

Such a surface is called a $K3$ surface ([12]). The Picard number ρ of X is defined by

$$\rho=\dim_{\boldsymbol{Q}} NS(X\otimes\bar{k})\otimes_{\boldsymbol{Z}} \boldsymbol{Q},$$

where $\bar{k}$ is the algebraic closure of k, and $NS(X\otimes\bar{k})$ is the Néron-Severi group of $X\otimes\bar{k}$. For any embedding of the field $\sigma: k\hookrightarrow \boldsymbol{C}$, put

$$\rho_\sigma=\dim_{\boldsymbol{Q}} NS(X\otimes_{k,\sigma} \boldsymbol{C})\otimes_{\boldsymbol{Z}} \boldsymbol{Q}.$$

Then the equality $\rho=\rho_\sigma$ holds.

The Betti numbers of X are given by

$$b_0=b_4=1, \quad b_1=b_3=0, \quad b_2=22.$$

Put $\rho_k=\dim_{\boldsymbol{Q}} NS(X)\otimes_{\boldsymbol{Z}} \boldsymbol{Q}$, and assume that $\rho_k=\rho$. We call $\lambda=b_2-\rho$ the Lefschetz number of X, which is the number of transcendental cycles independent modulo algebraic cycles.

Now let us recall the Brauer group $\mathrm{Br}(X\otimes\bar{k})$ of $X\otimes\bar{k}$. By Grothendieck [1], it is known to be a torsion group, and the Tate module $T_\ell(\mathrm{Br}(X\otimes\bar{k}))$ is given by the exact sequence of $\mathrm{Gal}(\bar{k}/k)$-modules

$$0\longrightarrow NS(X)\otimes \boldsymbol{Z}_\ell \longrightarrow H^2_{\text{ét}}(X\otimes\bar{k}, \boldsymbol{Z}_\ell[1])\longrightarrow T_\ell(\mathrm{Br}(X\otimes\bar{k}))\longrightarrow 0.$$

Here $\boldsymbol{Z}_\ell[1]$ is the Tate twist.

Put $V_\ell=T_\ell\otimes_{\boldsymbol{Z}_\ell} \boldsymbol{Q}_\ell$. The intersection form on $H^2_{\text{ét}}(X\otimes\bar{k}, \boldsymbol{Q}_\ell)$ is a symmetric bilinear form with values in $\boldsymbol{Q}_\ell[-2]$. We denote by $V_\ell(T)$ the orthogonal complement of $NS(X)\otimes_{\boldsymbol{Z}} \boldsymbol{Q}_\ell[-1]$. Then the restriction of the intersection form to $V_\ell(T)$ defines a non-degenerate bilinear form with values in $\boldsymbol{Q}_\ell[-2]$, and the above exact sequence induces an isomorphism of the ℓ-adic representations of $\mathrm{Gal}(\bar{k}/k)$:

$$V_\ell(T)[1] \xrightarrow{\sim} V_\ell(\mathrm{Br}(X\otimes\bar{k})).$$

Let us consider the ℓ-adic representation

$$\rho_{T,\ell}: \mathrm{Gal}(\bar{k}/k)\longrightarrow \mathrm{Aut}(V_\ell(T)).$$

By definition, $\lambda=b_2-\rho=\dim_{\boldsymbol{Q}_\ell} V_\ell(T)$. Since the characteristic of k is zero, by Hodge theory and by the Lefschetz criterion of algebraic cycles, we have $\lambda \geqq 2$ i.e. $\rho \leqq 20$.

To investigate $\rho_{T,\ell}$, we need the Kuga-Satake abelian varieties attached to $K3$ surfaces.

Choose an embedding $\sigma: k \hookrightarrow \boldsymbol{C}$. Then we denote by $H^2_\sigma(X, \boldsymbol{Z})$ the second Betti cohomology group of the complex analytic surface $(X \otimes_{k,\sigma} \boldsymbol{C})^{an}$. Fix an ample invertible sheaf L on X, and let $\Lambda \in H^2_\sigma(X, \boldsymbol{Z})$ be the Chern class of L. Let $P_\sigma(X)$ be the orthogonal complement of $\boldsymbol{Z}\Lambda$ in $H^2_\sigma(X, \boldsymbol{Z})$, with respect to the intersection from on $H^2_\sigma(X, \boldsymbol{Z})$. Let $C_+(P_\sigma(X))$ be the even Clifford algebra associated with the bilinear form on $P_\sigma(X)$, which is the restriction of the intersection form. Then Kuga-Satake [2] defined a structure of an abelian variety of dimension 2^{19} on the real torus $C_+(P_\sigma(X)) \otimes_{\boldsymbol{Z}} \boldsymbol{R} / C_+(P_\sigma(X))$, which we denote by $A_\sigma(X, L)$ or simply by $A_\sigma(X)$.

In the proof of [4], Deligne proved the following results:

Theorem (cf. Proposition (6.5) of [4]). *Let X be a $K3$ surface over k. Then the abelian variety $A_\sigma(X)$ has a model A defined over a finite extension k' of k. Moreover there are a Z-algebra $C = C_+(P_\sigma(X))$, an injection of algebras: $C \to \mathrm{End}_{k'}(A)$, and an isomorphism of ℓ-adic representations of* Gal $(\bar{k}/k')$

$$C_+(P(X, \boldsymbol{Q}_\ell)[1]) \xrightarrow{\sim} \mathrm{End}_C (H^1_{\text{ét}}(A \otimes \bar{k}, \boldsymbol{Q}_\ell)).$$

Here $P(X, \boldsymbol{Q}_\ell)$ is the orthogonal complement of $\boldsymbol{Z}_\ell \Lambda$ in $H^2_{\text{ét}}(X \otimes \bar{k}, \boldsymbol{Q}_\ell)$ with respect to ℓ-adic intersection form, and $C_+(P(X, \boldsymbol{Q}_\ell)[1])$ is the even Clifford algebra of $P(X, \boldsymbol{Q}_\ell)[1]$.

Let S be the image of $NS(X \otimes_{k,\sigma} \boldsymbol{C})$ in $H^2_\sigma(X, \boldsymbol{Z})$, and let T be the orthogonal complement of S in $H^2_\sigma(X, \boldsymbol{Z})$ with respect to the intersection form on $H^2_\sigma(X, \boldsymbol{Z})$. Clearly $\mathrm{rank}_{\boldsymbol{Z}} T = \lambda$. Recall that the intersection form on $H^2_\sigma(X, \boldsymbol{Z})$ has the signature $(3+, 19-)$, and that the restriction of this form to S has the signature $(1+, (\rho-1)-)$ by the index theorem of Hodge (cf. [12]). Therefore the restriction to T of the intersection form defines a non-degenerated symmetric bilinear form of the signature $(2+, (\lambda-2)-)$ on T. T is naturally equipped with the homogeneous Hodge structure of type $\{(2,0), (1,1), (0,2)\}$. And if we put $T_{\boldsymbol{C}} = T \otimes_{\boldsymbol{Z}} \boldsymbol{C}$, we have

$$\dim_{\boldsymbol{C}} T_{\boldsymbol{C}}^{2,0} = \dim_{\boldsymbol{C}} T_{\boldsymbol{C}}^{0,2} = 1 \quad \text{and} \quad \dim_{\boldsymbol{C}} T_{\boldsymbol{C}}^{1,1} = \lambda - 2.$$

By Satake [3], or by [4], we can define a structure of an abelian variety of dimension $2^{\lambda-2}$ on the real torus $C_+(T) \otimes_{\boldsymbol{Z}} \boldsymbol{R} / C_+(T)$. Here $C_+(T)$ is the even Clifford algebra associated with the intersection form on T. We denote this abelian variety by $A^T_\sigma(X)$.

Note that, as shown in the end of [3], the abelian variety $A_\sigma(X)$ is isogenous to the product of $2^{\rho-1}$ copies of $A^T_\sigma(X)$, and that the endomorphism algebra End $(A^T_\sigma(X)) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ is isomorphic to $C_+(T) \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$. Since

$A_\sigma(X)$ is defined over an algebraic number field k', $A_\sigma^T(X)$ is also defined over an algebraic number field k'', which is a finite extension of k'.

Remark. If we consider the totality of $K3$ surfaces with a fixed sublattice S of algebraic cycles in $H_\sigma^2(X, \boldsymbol{Z})$, in place of all polarized $K3$ surfaces, the methods of Deligne [4] are applicable to $A_\sigma^T(X)$, too. So we can obtain an analogy of Theorem for $A_\sigma^T(X)$ directly.

(A) *The Case* $\rho=20$. $\lambda=2$ in this case. Therefore, $A_\sigma^T(X)$ is one dimensional abelian variety, and $C_+(T)\otimes_Z \boldsymbol{Q}$ is an imaginary quadratic field. Accordingly, $A_\sigma^T(X)$ is an elliptic curve with complex multiplication. For abbreviation of notation, we denote this elliptic curve by E. E is defined over a certain algebraic number field k' which is a finite extension of k. There is a natural monomorphism of Gal $(\bar{k}/k')$-modules:

$$C_+(T)\otimes_Z \boldsymbol{Q}_\ell[-1] \longrightarrow H_{\text{ét}}^1(E\otimes\bar{k}, \boldsymbol{Q}_\ell)\otimes H_{\text{ét}}^1(E\otimes\bar{k}, \boldsymbol{Q}_\ell).$$

Denote the cokernel of this monomorphism by $V_\ell(T')$. Then, by Theorem, we have an isomorphism of Gal $(\bar{k}, k')$-modules

$$V_\ell(T) \xrightarrow{\sim} V_\ell(T').$$

Hence the restriction of the representation $\rho_{T,\ell}$ to Gal $(\bar{k}/k')$ is an abelian ℓ-adic representation, and for any finite extension k'' of k', the invariant part $V_\ell(T)[1]^{\mathrm{Gal}(\bar{k}/k'')}$ of the twist of $V_\ell(T)$ is reduced to zero, as conjectured by Tate [5].

Remark. Another method is given in the section 6 of Shioda-Inose [6], which depends on the more precise study of $K3$ surfaces with $\rho=20$.

Remark. We can obtain a more precise result. Namely we can show that $\rho_{T,\ell}$ is an abelian ℓ-adic representation of Gal $(\bar{k}/k)$. It follows from the intersection form on rank 2 $\boldsymbol{Q}_\ell$-module $V_\ell(T)$, and from the comparison theorem of Artin [7].

(B) *The Case* $\rho=19$. Let S be the sublattice $H_\sigma^2(X, \boldsymbol{Z})$ generated by the algebraic cycles on X. The restriction of the intersection form to S defines a symmetric bilinear form of signature $(1+, 18-)$. We denote by Δ_S the determinant of the matrix representing this bilinear form. Δ_S defines an element of $\boldsymbol{Q}^*/(\boldsymbol{Q}^*)^2$. Two cases occur, according as Δ_S is a square of a rational number or not.

(B.1) *The Case*. $\Delta_S=$ *square*, i.e. $\Delta_S \in (\boldsymbol{Q}^*)^2$. By the Poincaré duality, the intersection form is unimodular, and its signature is $(3+, 19-)$. Therefore we have

$$\Delta_S \cdot \Delta_T \equiv -1 \mod (\boldsymbol{Q}^*)^2.$$

In this case the quaternion algebra $C_+(T)\otimes_Z \boldsymbol{Q}$ over $\boldsymbol{Q}$ is isomorphic to the matrix algebra $M_2(\boldsymbol{Q})$. Thus, by Satake [3], $A_\sigma^T(X)$ is isogenous to the product $E\times E$ of an elliptic curve E without complex multiplication.*) E is defined over a certain finite extension k' of k. By Theorem, we have an isomorphism of Gal $(\bar{k}/k')$-modules:

$$V_\ell(T) \xrightarrow{\sim} \mathrm{Symm}^2\, H^1_{\text{ét}}(E\otimes\bar{k}, \boldsymbol{Q}_\ell).$$

Here Symm^2 means the symmetric tensor product of degree 2. By the results of Serre [8], [9], for any finite extension k'' of k', the invariant part $V_\ell(T)[1]^{\mathrm{Gal}(\bar{k}/k'')}$ is zero. Thus we have verified the Tate conjecture.

(B.2) *The Case* $\Delta_S \notin (\boldsymbol{Q}^*)^2$. In this case, the quaternion algebra $C_+(T)\otimes_Z \boldsymbol{Q}$ is an indefinite division algebra. Thus, by [3], $A^T_\sigma(X)$ is a simple abelian variety of dimension two. It is known that the rank of the Néron-Severi group $A^T_\sigma(X)$ is three. By Theorem, $A^T_\sigma(X)$ has a model A defined over a certain finite extension k' of k, and we have an isomorphism of Gal $(\bar{k}/k')$:

$$V_\ell(T) \xrightarrow{\sim} H^2_{\text{ét}}(A\otimes\bar{k}, \boldsymbol{Q}_\ell)/(NS(A\otimes\bar{k})\otimes_Z \boldsymbol{Q}[-1]).$$

By the results of Ohta [10] on the ℓ-adic representation on $H^1_{\text{ét}}(A\otimes\bar{k}, \boldsymbol{Q}_\ell)$ of a so-called "false elliptic curve" A, we can readily check that the invariant part $V_\ell(T)[1]^{\mathrm{Gal}(\bar{k}/k'')}$ is reduced to zero for any finite extension k'' of k. Thus the Tate conjecture is true in this case too.

(C) *A few words for the case* $\rho=18$. This case is also divided into two cases, according as Δ_T is a square or not.

(C.1) When Δ_T is a square of a rational number, $A^T_\sigma(X)$ is isogenous to the product $E_1\times E_1\times E_2\times E_2$ by [3]. Here E_1 and E_2 are non-isogenous two elliptic curves. For a certain finite extension k' of k containing the fields of definition of E_1 and E_2, we have an isomorphism:

$$V_\ell(T) \xrightarrow{\sim} H^1_{\text{ét}}(E_1\otimes\bar{k}, \boldsymbol{Q}_\ell)\otimes H^1_{\text{ét}}(E_2\otimes\bar{k}, \boldsymbol{Q}_\ell)$$

of the ℓ-adic representations of Gal $(\bar{k}/k')$.

Thus the validity of the Tate conjecture in this case is equivalent to the following:

For any finite extension k'' *of* k', *the* Gal $(\bar{k}/k'')$-*module* $H^1_{\text{ét}}(E_1\otimes\bar{k}, \boldsymbol{Q}_\ell)$ *is not isomorphic to* $H^1_{\text{ét}}(E_2\otimes\bar{k}, \boldsymbol{Q}_\ell)$.

This seems to be still an open problem for general E_1, E_2. See the end of [9].

(C.2) When Δ_T is not square, $\boldsymbol{Q}(\sqrt{\Delta_T})$ is a real quadratic field, because Δ_T is positive. In this case, $C_+(T)\otimes_Z \boldsymbol{Q}$ is a matrix algebra $M_2(\boldsymbol{Q}(\sqrt{\Delta_T}))$. By [3], $A^T_\sigma(X)$ is isogenous to the product $A\times A$ of a two dimensional abelian variety A with real multiplication $\boldsymbol{Q}(\sqrt{\Delta_T})$. We can find a sufficient large finite extension k' of k, such that A is defined over k' and such that the rank of $NS(A\otimes k')$ is equal to 2. Hence Theorem implies an isomorphism:

$$V_\ell(T) \xrightarrow{\sim} H^2_{\text{ét}}(A\otimes\bar{k}, \boldsymbol{Q}_\ell)/NS(A\otimes k')\otimes\boldsymbol{Q}_\ell[-1]$$

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*) The fact that $A_\sigma(X)$ is isogenous to the product of 2^{19} copies of an elliptic curve was suggested by Kuga several years ago. I would like to thank him.

of the ℓ-adic representations of Gal $(\bar{k}/k')$. Unfortunately the results of Ribet [11] is insufficient to assure the Tate conjecture without irrelevant conditions.

We can treat the case $\rho\leqq 17$ similarly. But for such case, it seems that the necessary results for ℓ-adic representation of the corresponding abelian varieties is not known.

As a generalization of [6], we can expect the following fact (open problem).

For any K3 surface X with $\rho\geqq 18$, *or with* $\rho=17$ *and* $\Delta_S\in(\boldsymbol{Q}^*)^2$, *there exist an abelian variety A of dimension two and an algebraic correspondence* ψ *of X to the Kummer surface* Km(A), *such that* ψ *induces an isomorphism of* $V_\ell(\mathrm{Br}(X\otimes\bar{k}))$ *to* $V_\ell(\mathrm{Br}(A\otimes\bar{k}))$ *and an isomorphism (with Hodge structure of)* $T\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ *to {the module of transcendental cycles of A}*$\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$.

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