67. *Nonexistence of Minimizing Harmonic Maps from 2-Spheres*

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§1. **Introduction.** Let (M, g) and (N, h) be compact Riemannian manifolds and $C^\infty(N, M)$ be the space of all smooth maps from N to M with the C^∞ topology. For $f \in C^\infty(N, M)$ we define its *energy* $E(f)$ by

$$E(f)=\frac{1}{2}\int_N h^{ij}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}g_{\alpha\beta}*1. \tag{1.1}$$

A *harmonic map* is, by definition, a critical point of the functional E. A harmonic map is said to be *minimizing* if it minimizes energy in its connected component of $C^\infty(N, M)$, i.e. in its homotopy class.

When $\dim N=1$, N is a circle S^1 and a harmonic map $f: S^1 \to M$ is a closed geodesic. It is well known that every component of $C^\infty(S^1, M)$ contains a minimizing closed geodesic. In contrast with this, when $\dim N=2$, it is not always true that there exists a minimizing harmonic map in each component of $C^\infty(N, M)$. For instance there exists no minimizing harmonic map of degree ± 1 from a Riemann surface of genus $\geqq 1$ to a Riemann sphere whatever metrics are chosen on them ([5]).

On the other hand, Sacks and Uhlenbeck [8] established an existence result when $N=S^2$. Their result was applied to the proof of Frankel's conjecture by Siu and Yau [9] and to the study on the topology of 3-manifolds by Meeks and Yau [7]. The following is a result of Sacks and Uhlenbeck refined by Siu and Yau. Let M be a compact 1-connected Riemannian manifold. Let $f_0 \in C^\infty(S^2, M)$. Then there exist minimizing harmonic maps $f_1, \cdots, f_k \in C^\infty(S^2, M)$ such that $\sum_{i=1}^k f_i = f_0$ in $\pi_2(M)$ and that

$$\sum_{i=1}^k E(f_i)=\inf\left\{\sum_{i=1}^p E(g_i)\,|\, p \in N,\ \sum_{i=1}^p g_i = f_0 \text{ in } \pi_2(M)\right\}. \tag{1.2}$$

However it has been unknown whether one can always find a single minimizing harmonic map homotopic to f_0 or not.

The purpose of this paper is to give a Riemannian manifold M and a component of $C^\infty(S^2, M)$ such that no minimizing harmonic map exists in this component.

§2. **Statement of the result. Theorem.** *Let M be a compact 1-connected Kähler surface. Suppose there are two disjoint rational* *curves C and D in M such that $(C+D)\circ C<0$ and $(C+D)\circ D\neq 0$. Then the homotopy class of $C+D$ contains no minimizing harmonic map.*

We can easily construct such curves C and D in Theorem as follows. Let M' be a compact Kähler surface which contains a rational curve D such that $D\circ D\neq 0$. Choose any $p\in M'-D$ and let $\pi: M\to M'$ be the blow-up of M' at p. If we set $C=\pi^{-1}(p)$ and $D=\pi^{-1}(D)$, then $(C+D)\circ C=-1$ and $(C+D)\circ D\neq 0$.

§3. **Properties of energy and area.** Let (M, g) and (N, h) be compact Riemannian manifolds. Throughout the rest of this paper we assume that N is 2-dimensional.

For $f\in C^\infty(N, M)$ we define its *area* $Area(f)$ by

$$Area(f)=\int_N \sqrt{\det\left(g_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\right)}dx^1\wedge dx^2, \tag{3.1}$$

where (x^1, x^2) is a local coordinate system on N. The following lemma is clear from the relation between arithmetic mean and geometric mean.

Lemma 3.1.

$$Area(f)\leqq E(f). \tag{3.2}$$

The equality holds iff f is weakly conformal, i.e. there exists a non-negative smooth function ρ on N such that $f^*g=\rho h$.

Next we consider the case when M and N are compact Kähler manifolds. For $f\in C^\infty(N, M)$ we define $E'(f)$ and $E''(f)$ by

$$E'(f)=\int_N g_{\alpha\beta}\frac{\partial f^\alpha}{\partial w}\overline{\frac{\partial f^\beta}{\partial w}}\sqrt{-1}dw\wedge d\overline{w} \quad \text{and} \tag{3.3}$$

$$E''(f)=\int_N g_{\alpha\beta}\frac{\partial f^\alpha}{\partial \overline{w}}\overline{\frac{\partial f^\beta}{\partial \overline{w}}}\sqrt{-1}dw\wedge d\overline{w},$$

where w is a holomorphic local coordinate on N. Then we have

Lemma 3.2 ([6]).

$$E(f)=E'(f)+E''(f) \quad \text{and} \quad \int_N f^*\omega=E'(f)-E''(f), \tag{3.4}$$

where $\omega=\sqrt{-1}g_{\alpha\beta}dz^\alpha\wedge d\bar{z}^\beta$ is the Kähler form of M.

§4. **Complex analyticity of certain harmonic maps.**

Lemma 4.1. *Let M be a 1-connected Riemannian manifold of dimension $\geqq 3$. Let ϕ and ψ be any smooth map from S^2 to M. Suppose there exists a minimizing harmonic map from S^2 to M such that $f=\phi+\psi$ in $\pi_2(M)$, then we have*

$$E(f)\leqq E(\phi)+E(\psi). \tag{4.1}$$

Proof. We consider S^2 as $\boldsymbol{R}^2\cup\{\infty\}$. Let g_0 be a piecewise smooth map from S^2 to M parametrizing ϕ, a curve between $\phi(\infty)$ and $\psi(0)$, and ψ. Then $g_0=\phi+\psi$ in $\pi_2(M)$ and

$$Area(g_0)=Area(\phi)+Area(\psi). \tag{4.2}$$

Let ε be any small positive number. We can approximate g_0 by a smooth immersion g_1 such that

$$(4.3)\qquad Area(g_1) \leqq Area(g_0)+\varepsilon.$$

There exists a diffeomorphism of S^2 pulling back the conformal structure defined by $g_1^* ds_M^2$ to the standard conformal structure. We define g_2 to be g_1 composed with this diffeomorphism so that g_2 is conformal. Then we obtain from Lemm 3.1

$$(4.4)\qquad \begin{aligned} E(f) &\leqq E(g_2) = Area(g_2) = Area(g_1) \\ &\leqq Area(\phi) + Area(\psi) + \varepsilon \\ &\leqq E(\phi) + E(\psi) + \varepsilon. \end{aligned}$$

Since ε is arbitrary, our conclusion follows.

Lemma 4.2. *Let M be a 1-connected Kähler manifold. Let ϕ and ψ be holomorphic maps from 1-dimensional complex projective space $\boldsymbol{P}$ with the Fubini-Study metric to M. Then every minimizing harmonic map f from $\boldsymbol{P}$ to M such that $f=\phi+\psi$ in $\pi_2(M)$ is holomorphic.*

Proof. Let ω be the Kähler from of M. By Lemma 4.1 we have

$$(4.5)\qquad \int_P f^*\omega = \int_P \phi^*\omega + \int_P \psi^*\omega = E(\phi) + E(\psi) \geqq E(f).$$

It follows from (4.5) and Lemma 3.2 that $E''(f)=0$ and so f is holomorphic.

§ 5. Proof of Theorem. Suppose there were a minimizing harmonic map $f: \boldsymbol{P} \to M$ in the homotopy class of $C+D$. Then f is holomorphic by Lemma 4.2. Since f meets both C and D and $C \cap D = \phi$, f meets C at finite points. Hence f intersects C positively. This contradicts our assumption, completing the proof.

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