66. A Note on the Large Sieve. IV

By Yoichi MOTOHASHI
Department of Mathematics, College of Science
and Technology, Nihon University

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1. The purpose of the present note is to show a hybrid of the multiplicative large sieve and the Rosser-Iwaniec linear sieve.

We retain most of the notations of our preceding paper [6], and in addition we introduce the following conventions: Let χ be a Dirichlet character, and put

$$S(A, z, \chi)=\sum_{\substack{n\in A\\(n,P(z))=1}}\chi(n)a_n,$$

where a_n are arbitrary complex numbers. We put also, for $\chi \pmod q$,

$$R_d(\chi)=\sum_{\substack{n\in A\\n\equiv 0\,(\mathrm{mod}\,d)}}\chi(n)-\varepsilon_\chi|\chi(d)|\frac{\delta(d)}{d}\prod_{p\mid q}\left(1-\frac{\delta(p)}{p}\right)X,$$

in which ε_χ is 1 if χ is principal, and 0 otherwise.

Then our hybrid sieve is

Theorem 1. *Let Δ be a finite set of primitive Dirichlet characters, and let M, N be arbitrary but $MN\geq z^2$. Then we have, as $z\to\infty$,*

$$\sum_{\chi\in\Delta}|S(A,z,\chi)|^2$$
$$\leq\left[XV(z)\left\{F\left(\frac{\log MN}{\log z}\right)+o(1)\right\}+O(E)\right]\sum_{\substack{n\in A\\(n,P(z))=1}}|a_n|^2,$$

where

$$E=\max_{\alpha,\beta}\max_{\psi\in\Delta}\sum_{\chi\in\Delta}\Big|\sum_{\substack{m<M\\n<N}}\alpha_m\beta_nR_{mn}(\chi\bar{\psi})\Big|,$$

$\{\alpha_m\}$, $\{\beta_n\}$ *being variable vectors such that* $|\alpha_m|\leq 1$, $|\beta_n|\leq 1$.

The proof which will be given in [7] is a direct application of Iwaniec's important idea [2] to the dual form

$$\sum_{\substack{n\in A\\(n,P(z))=1}}\Big|\sum_{\chi\in\Delta}\chi(n)b_\chi\Big|^2,$$

where b_χ are arbitrary complex numbers.

2. To illustrate the power of the above theorem we prove briefly the following result of the Brun-Tichmarsh type:

Theorem 2. *If $x\geq k^2Q^4\to\infty$, then we have*

$$\sum_{\substack{q\leq Q\\(q,k)=1}}\sum_{\chi\,(\mathrm{mod}\,q)}^{*}\Big|\sum_{\substack{p\equiv l\,(\mathrm{mod}\,k)\\p<x}}\chi(p)\Big|^2$$
$$\leq(2+o(1))x\left(\varphi(k)\log\left(\frac{x}{Q\sqrt{k}}\right)\right)^{-1}\pi(x;k,l),$$

where $\sum^*$ denotes a sum over primitive characters.

This is a large sieve extension of a result of Iwaniec [2, Theorem 3], and at the same time an improvement upon a result of [4] the first paper of this series (see also [5]).

For the proof we set in Theorem 1 $A=\{n\,;\,n\equiv l \pmod{k},\ n<x\}$, $P=\{p\,;\,p\nmid k\}$, $z=(MN)^{1/3}$, $\Delta=\{\chi \text{ primitive} \pmod{q}\,;\,q\le Q,\ (q,k)=1\}$, and $a_n=1$ if n is a prime and $a_n=0$ otherwise. Then $\delta(d)=1$ for $d\,|\,P(z)$, and $X=x/k$. So our problem is now the estimation of E. For this sake we put

$$A(s,\chi)=\sum_{m<M}\alpha_m\chi(m)m^{-s},\qquad B(s,\chi)=\sum_{n<N}\beta_n\chi(n)n^{-s}.$$

Using Perron's inversion formula we get, for $\chi \pmod{q}$ and $T\ge 1$,

$$\sum_{\substack{m<M\\ n<N}}\alpha_m\beta_n R_{mn}(\chi)$$
$$=\frac{1}{2\pi i\varphi(k)}\sum_{\xi \,(\mathrm{mod}\, k)}\bar{\xi}(l)\int_{1/2-iT}^{1/2+iT}L(s,\chi\xi)A(s,\chi\xi)B(s,\chi\xi)\frac{x^s}{s}ds$$
$$+O\left\{\left(\left(\frac{xMNQk}{T}\right)^{1/2}+\frac{x}{T}\right)(\log xMNQk)^3\right\}.$$

Hence setting $T=(xMNkQ)^c$ with a sufficiently large c we have

$$E\ll\frac{x^{1/2}}{\varphi(k)}(\log xMNQk)\max_{\alpha,\beta}\max_{\psi\in\Delta}\max_{1\le U\le T}U^{-1}$$
$$\times\left\{\int\sum_{\xi \,(\mathrm{mod}\, k)}\sum_{\chi\in\Delta}|L(s,\chi\psi\xi)|^4|ds|\right\}^{1/4}\left\{\int\sum_{\xi \,(\mathrm{mod}\, k)}\sum_{\chi\in\Delta}|A(s,\chi\psi\xi)|^4|ds|\right\}^{1/4}$$
$$\times\left\{\int\sum_{\xi \,(\mathrm{mod}\, k)}\sum_{\chi\in\Delta}|B(s,\chi\psi\xi)|^2|ds|\right\}^{1/2},$$

where the integrations are all along the straight line $[1/2-iU,1/2+iU]$. By a simple application of the multiplicative large sieve we see that the second and the third integrals are, respectively,

$$O\{(M^2+kQ^2U)\log^3 M\}\quad\text{and}\quad O\{(N+kQ^2U)\log N\}.$$

On the other hand the method of Ramachandra (cf. [1, pp. 80–81]) yields

$$\int\sum_{\xi \,(\mathrm{mod}\, k)}\sum_{\chi\in\Delta}|L(s,\chi\psi\xi)|^4|ds|\ll(kQ^2U)^{1+\varepsilon}.$$

Thus

$$E\ll\frac{1}{\varphi(k)}(xQ\sqrt{k})^{1/2}(M^2+kQ^2)^{1/4}(N+kQ^2)^{1/2}(xMNQk)^{\varepsilon}.$$

This implies that an optimal choice of M,N is given by $N=M^2\ge kQ^2$, $M=(x/(\sqrt{k}\,Q))^{1/3-\varepsilon}$. And after some additional considerations about the primes ≥ 2 we conclude the proof of Theorem 2.

It should be remarked that Iwaniec [3] has given various methods to deal with E when Δ consists of only the trivial character, and most of his arguments may be carried into the more general situation of the present note. Thus in particular Theorem 2 is by no means the best result deducible from Theorem 1; the detailed discussions will be given in [7].

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References

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