

65. On the Linear Sieve. I

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1. Two different proofs of the linear sieve are known: One is due to Jurkat-Richert [6] and the other to Rosser (unpublished, but see [8]) and Iwaniec [2] (see also [3]). Comparing these proofs one may note that Jurkat-Richert's procedure is simpler than that of Iwaniec in the treatment of the convergence problem arising from the infinite iteration of the Buchstab identity. But the Rosser-Iwaniec sieve has the important advantage that it admits a very flexible bilinear form for the error-term; this was discovered by Iwaniec [4] and must be a milestone in the sieve history as its applications (cf. [5]) indicates clearly. It seems unlikely, however, that the similar improvement may be introduced to the Jurkat-Richert sieve; the reason for this lies in their use of the Selberg sieve as an aid.

Now the purpose of this note is to show briefly that one may reduce considerably the aforementioned difficulty in the Rosser-Iwaniec sieve by combining an important idea of Jurkat-Richert [6] with the Rosser truncation of the Buchstab identity.*) The details will be given in [7], and here we indicate only the clues.

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2. Now let A be a finite sequence of integers and P a set of primes. Let $S(A, z) = |\{a \in A \mid (a, P(z)) = 1\}|$, where $P(z) = \prod p$ over $p < z$, $p \in P$. Let $A_d = \{a \in A \mid a \equiv 0 \pmod{d}\}$ and put $R_d = |A_d| - X\delta(d)/d$, where X is a parameter and δ a multiplicative function. As in [1] we introduce the condition $\Omega_2(1, L)$: For any $2 \leq u \leq v$

$$-L \leq \sum_{\substack{u \leq p \leq v \\ p \in P}} \frac{\delta(p)}{p} \log p - \log \frac{w}{v} \leq C,$$

where L is a parameter and C a constant. Next we define functions F and f by $F(u) = 2e^\gamma/u$, $f(u) = 0$ if $0 < u \leq 2$ and by $(u f(u))' = F(u-1)$, $(u F(u))' = f(u-1)$ if $u > 2$, where γ is the Euler constant; also we put $\phi_\nu(u) = F(u)$ if ν is odd, and $\phi_\nu(u) = f(u)$ if ν is even. Finally we denote by $E(y)$ the sum $\sum |R_d|$ over $d < y$, $d \mid P(z)$, y being another parameter.

*) It seems that this confirms partially Selberg's anticipation expressed at the bottom lines of [8, p. 343].

Then the fundamental theorem in the linear sieve theory is the estimate : For $2 \leq z \leq y$

$$(-1)^{\nu-1} \left\{ S(A, z) - XV(z) \phi_{\nu} \left(\frac{\log y}{\log z} \right) \right\} \leq LXV(z) (\log y)^{-\alpha} + E(y),$$

where α is a positive absolute constant, and $V(z) = \prod (1 - \delta(p)/p)$ over $p < z, p \in P$.

3. Next we introduce another parameter $z_1, 2 \leq z_1 \leq z$. And for $\nu = 0, 1$ we define $\rho_{\nu}(d)$ to be the characteristic function of the set of integers d such that $d = p_r p_{r-1} \cdots p_1$ with $p_j \in P (1 \leq j \leq r), z_1 \leq p_r < \cdots < p_1 < z$ and $p_{2k+\nu}^3 p_{2k+\nu-1} \cdots p_2 p_1 < y (1 \leq 2k + \nu \leq r)$. Then the Rosser truncation of the Buchstab identity gives

$$(-1)^{\nu} S(A, z) \geq (-1)^{\nu} \sum_d \mu(d) \rho_{\nu}(d) S(A_d, z_1),$$

where μ is the Möbius function. Then we set $z_1 = \exp((\log y)^{7/10})$ and we apply Brun's pure sieve (cf. [1, p. 46]) to every $S(A_d, z_1)$; we get

$$\left| S(A_d, z_1) - X \frac{\delta(d)}{d} V(z_1) (1 + O((\log y)^{-3})) \right| \leq \sum_g |R_{dg}|$$

where $g|P(z_1), g < z_0 = \exp(10(\log y)^{7/10} \log \log y)$. Hence we have to compute the sum

$$T_{\nu} = V(z_1) \sum_d \frac{\mu(d)}{d} \rho_{\nu}(d) \delta(d).$$

For this sake we modify Theorem 8.2 of [1, p. 229] as follows :

Lemma 1. *Provided $\Omega_2(1, L)$ and $2 \leq z_1 \leq z \leq y^{(1+\nu)/2} (\nu = 0, 1)$, we have*

$$\begin{aligned} V(z) \phi_{\nu} \left(\frac{\log y}{\log z} \right) &= \sum_d d^{-1} \mu(d) \rho_{\nu}(d) \delta(d) V(z_1) \phi_{\nu+\omega(d)} \left(\frac{\log y/d}{\log z_1} \right) \\ &\quad + O(LV(z) (\log y)^{-1/10}). \end{aligned}$$

Thus noticing $\phi_{\nu}(u) = 1 + O(e^{-u})$ we get

$$\begin{aligned} T_{\nu} - V(z) \phi_{\nu} \left(\frac{\log y}{\log z} \right) &\ll \sum_d d^{-1} \rho_{\nu}(d) \delta(d) V(z_1) \exp \left(-\frac{\log y/d}{\log z_1} \right) \\ &\quad + LV(z) (\log y)^{-1/10}. \end{aligned}$$

The last sum is divided into two parts \sum_{I} and \sum_{II} according to $\omega(d) < 2B$ and $2B \leq \omega(d)$ respectively, where $\omega(d)$ is the number of prime factors of d , and B satisfies $3^B = (\log y)(3 \log z \log \log y)^{-1}$. To estimate \sum_{I} we note that $\rho_{\nu}(d) = 1$ implies $\log(y/d) > (1/3)^{\omega(d)/2} \log y$ which can be shown as (4.11) of [1, p. 233]. The estimation of \sum_{II} is more difficult. We note first that if $\rho_{\nu}(d) = 1$ then

$$V(z_1) \exp \left(-\frac{\log y/d}{\log z_1} \right) \ll V(p(d)) \exp \left(\frac{\log y/d}{\log p(d)} \right)$$

where $p(d)$ is the least prime factor of d . And then we appeal to the crucial

Lemma 2. *Provided $\Omega_2(1, L)$ and $2 \leq z_1 \leq z \leq y^{1/2}$, we have*

$$\sum_{\substack{z_1 \leq p < q < z \\ p^2 q < y}} \frac{\delta(pq)}{pq} V(p) \exp\left(-\frac{\log y/pq}{\log p}\right) \\ \leq \varepsilon V(z) \exp\left(-\frac{\log y}{\log z}\right) \{1 + O(L(\log y)^{-2/5})\},$$

where $\varepsilon = 1/2(1/3 + \log 3)$.

This corresponds to Lemma 8.2 of [1, p. 229], but the proof requires more care. Applying this [b/2] times we get

$$\sum_{\omega(d)=b} \frac{\rho_v(d)\delta(d)}{d} V(p(d)) \exp\left(-\frac{\log y/d}{\log p(d)}\right) \ll \left(\frac{4}{5}\right)^{b/2} V(z).$$

Collecting these consideration we obtain the fundamental theorem, but with $E(z_0 y)$ instead of $E(y)$; this blemish is easy to be removed.

Finally it may be worth remarking that our argument is even simpler than that of Jurkat-Richert, because we do not have anything corresponding to the last sum of (4.6) of [1, p. 232].

Added in Proof. After submitting the paper the author found that in a slightly different context Iwaniec ([2, Lemma 4]) had obtained a result essentially same as Lemma 2 above.

References

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