7. *Representation Groups of the Group* $Z_{p^n}\times Z_{p^n}$

By Kanzi SUZUKI
Department of Mathematics, Waseda University

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Introduction. The dihedral group D_2 and the quaternion group Q_2 of order 8 have the same character table (Feit [1, §§ 7 and 11]). Generally the two non-abelian groups of order p^3 (p a prime number) have the same character table (Brauer [3, § 4]). It is easily shown that these groups are characterized as the representation groups of the product $Z_p\times Z_p$ of cyclic groups of order p.

In this note, we consider the representation groups of $Z_{p^n}\times Z_{p^n}$, the product of cyclic groups of order p^n, and we deal with those complex characters. In § 1, we show that there exist two non-isomorphic representation groups of $Z_{p^n}\times Z_{p^n}$ (Theorem 1). When $n\geqq 2$, these groups have not the same character table (§ 3, Corollary 2), but have the conjugacy classes of the type described in Proposition 1. Their non-linear irreducible characters are constructed by the abelian residue groups of certain normal subgroups (Theorem 2).

1. Generators and relations. Let G be a finite group and C^* the multiplicative group of the complex number field C. When G acts trivially on C^*, the finite abelian group $H^2(G, C^*)$ is called the Schur multiplier of G. A group H is called a representation group of G when H has a central subgroup A such that 1) $H/A\cong G$, 2) $|A| =|H^2(G, C^*)|$ and 3) A is contained in the commutator subgroup $D(H)$.

Let H be a representation group of $Z_{p^n}\times Z_{p^n}$, where p is a prime number and n is a positive integer. The sequence

$$1\to A\to H\to Z_{p^n}\times Z_{p^n}\to 1$$

is exact, and $A=D(H)$ is contained in the center $Z(H)$ of H. We choose representatives t, r of inverse images of two generators of $Z_{p^n}\times Z_{p^n}$. Then A is the cyclic group generated by the commutator $s=t^{-1}rtr^{-1}$ of order p^n, because $H^2(Z_{p^n}\times Z_{p^n}, C^*)\cong Z_{p^n}$ (see Suzuki [2, p. 261]).

Consequently, the elements t, r and s generate H, i.e.,

$$H=\langle t, r, s\rangle \tag{1}$$

and satisfy the relations

$$r^{p^n},\quad t^{p^n}\in\langle s\rangle,\quad s^{p^n}=1 \tag{2}$$

$$ts=st,\quad rs=sr\quad\text{and}\quad t^{-1}rt=rs \tag{3}$$

where p^n is the least positive integer q such that $t^q\in\langle s\rangle$ (this p^n is also the least positive integer q such that $r^q\in\langle s\rangle$). Note that $A=Z(H)$.

It is clear that groups defined by the relations (1)–(3) are representation groups of $Z_{p^n}\times Z_{p^n}$.

Theorem 1. *There is only two non-isomorphic representation groups of* $Z_{p^n}\times Z_{p^n}$.

Proof. Let H be a representation group of $Z_{p^n}\times Z_{p^n}$ defined by the conditions (1)–(3). Since $\langle t, s\rangle$ and $\langle r, s\rangle$ are abelian groups of order p^{2n}, we may consider the following three cases:

$$H_1=\langle t, r, s\rangle\,;\ t^{p^n}=r^{p^n}=s^{p^n}=1,\ ts=st,\ rs=sr=t^{-1}rt$$
$$H_2=\langle t, r\rangle\,;\ t^{p^n}=r^{p^{2n}}=1,\ t^{-1}rt=r^{1+p^n},\ (r^{p^n}=s)$$
$$H_3=\langle t, r\rangle\,;\ r^{p^{2n}}=1,\ t^{p^n}=r^{p^n},\ t^{-1}rt=r^{1+p^n},\ (r^{p^n}=s).$$

If p is odd or if $p=2$, $n\geqq 2$, then H_2 is isomorphic to H_3, not isomorphic to H_1. If $p=2$, $n=1$, then $H_1=\langle t, rt\rangle$ is isomorphic to $H_2=D_2$ and is not isomorphic to $H_3=Q_2$.

2. Conjugacy classes. In the sequel, we denote by H a representation group of $Z_{p^n}\times Z_{p^n}$ with the conditions (1)–(3), and by $m(i, j)$ a non-negative integer such that $p^{m(i,j)}$ is the largest power of p dividing the greatest common divisor (i, j) of i and j.

Proposition 1. 1) *The number of conjugacy classes of H is given by*

$$p^{2n}+p^{n-1}(p^n-1).$$

2) *Each conjugacy class of H not consisting of central elements is of the form*

$$t^i r^j s^k\langle s^{p^{m(i,j)}}\rangle \tag{4}$$

where $0\leqq i, j\leqq p^n-1$ and $0\leqq k\leqq p^{m(i,j)}-1$, except the case $i=j=0$.

Proof of 2). Each element h of H is uniquely expressible in the form

$$h=t^i r^j s^{k_0},\quad 0\leqq i, j, k_0\leqq p^n-1.$$

If h is not contained in $Z(H)$, then i or j is a non-zero integer. Using the relation (3), we have

$$(t^u r^v)^{-1}h(t^u r^v)=hs^{uj-vi}. \tag{5}$$

If we put $(i, j)=qp^{m(i,j)}$ with $p\nmid q$, the relation (5) yields that the conjugacy class containing h is given by $t^i r^j s^k\langle s^{p^{m(i,j)}}\rangle$, where $k\equiv k_0 \bmod p^{m(i,j)}$ $(0\leqq k\leqq p^{m(i,j)}-1)$.

Proof of 1). For a non-negative integer m, we consider the number of pairs of i and j satisfying the conditions that $0\leqq i, j\leqq p^n-1$ and $m(i, j)=m$, except the case $i=j=0$. This number is given by

$$p^{n-m}\varphi(p^{n-m})+\varphi(p^{n-m})p^{n-m}-\varphi(p^{n-m})^2$$
$$=\varphi(p^{n-m})p^{n-m-1}(p+1)$$

where φ denotes the Euler function. Since k (resp. m) ranges over all non-negative integers less than p^m (resp. n), the number of the conjugacy classes of the form (4) is given by

$$\sum_{m=0}^{n-1} p^m\varphi(p^{n-m})p^{n-m-1}(p+1)=(p+1)p^{n-1}\sum_{m=0}^{n-1}\varphi(p^{n-m}).$$

On the other hand the number of conjugacy classes consisting of a single central element is $|Z(H)|=p^n$, so the number of all conjugacy classes of H is given by

$$p^n+(p+1)p^{n-1}\sum_{m=0}^{n-1}\varphi(p^{n-m})=p^{2n}+p^{n-1}(p^n-1).$$

3. Irreducible characters. We shall construct complex irreducible characters of H.

Lemma 1. *If a finite group G has an abelian normal subgroup N such that G/N is cyclic, then any irreducible character of G is induced from a linear character of a subgroup L containing N.*

Remark. This Lemma is also true even if G/N is abelian (see Yamada [4, Theorem 1]).

Proof. Let χ be an irreducible character of G and λ a linear constituent of the restriction $\chi|_N$ of χ to N. Let L be the inercia group of $\lambda: L=\{g\in G;\lambda^g=\lambda\}$, where $\lambda^g(h)=\lambda(g^{-1}hg)$ for any $h\in N$. By Clifford's theorem there exists an irreducible character θ of L such that $\chi=\theta^G$ (θ^G denotes the induced character) and $\theta|_N=e\lambda$ for some positive integer e. Since L/N is cyclic, we obtain $e=1$ (Feit [1, (9.12)]), which shows that θ is linear, and the proof is complete.

Now a representation group H has a maximal abelian normal subgroup $\langle r,s\rangle$. Since $H/\langle r,s\rangle$ is cyclic, we have the unique composition series over $\langle r,s\rangle$

$$\langle r,s\rangle=L_n\subset L_{n-1}\subset\cdots\subset L_1\subset H$$

where $L_m=\langle t^{p^m},r,s\rangle$. Since $|H/L_m|=p^m$, it follows from Lemma 1 that H has a non-linear irreducible character of degree p^m induced from a linear character λ of L_m $(1\leqq m\leqq n)$. Note that $\lambda(s)$ is a p^m-th root of unity, because $D(L_m)=\langle s^{p^m}\rangle$.

Lemma 2. *The induced character λ^H is irreducible if and only if $\lambda(s)$ is a primitive p^m-th root of unity.*

Proof. By Frobenius reciprocity theorem, we have

$$(\lambda^H,\lambda^H)=(\lambda^H|_{L_m},\lambda)=\sum_{u=0}^{p^m-1}(\lambda^{t^u},\lambda).$$

Hence λ^H is irreducible if and only if $\lambda^{t^u}\neq\lambda$ for all integer $u\not\equiv 0 \bmod p^m$. Since any element h of L_m has the form $h=t^{ip^m}r^js^k$ $(0\leqq i\leqq p^{n-m}-1,$ $0\leqq j,\ k\leqq p^n-1)$, it follows from (5) that

$$\lambda^{t^u}(h)=\lambda(h)\lambda(s)^{uj}\qquad(j=0,1,\cdots,p^n-1). \tag{6}$$

Therefore $\lambda^{t^u}\neq\lambda$ for all integer $u\not\equiv 0 \bmod p^m$ if and only if $\lambda(s)$ is a primitive p^m-th root of unity, which proves Lemma 2.

To determine irreducible characters of H, we define the normal subgroups

$$T_m=\langle t^{p^m},r^{p^m},s\rangle,\qquad(m=1,2,\cdots,n),$$

and put

$$\bar{T}_m=T_m/\langle s^{p^m}\rangle,\qquad(m=1,2,\cdots,n).$$

Then T_m is contained in L_m and $\bar{T}_m$ is abelian.

For each $h\in T_m$, we denote by $\bar{h}$ the image of h under the natural homomorphism $T_m\to\bar{T}_m$.

Theorem 2. *A class function χ is an irreducible character of H of degree p^m if and only if*

$$\chi(h)=\begin{cases}0, & (h\notin T_m)\\ p^m\bar{\mu}(\bar{h}), & (h\in T_m),\end{cases} \tag{7}$$

where $\bar{\mu}$ is a linear character of $\bar{T}_m$ whose restriction to $\langle\bar{s}\rangle$ is faithful.

Proof. Suppose that χ is an irreducible character of H of degree p^m, then χ is induced from a linear character λ of L_m and χ vanishes outside L_m. For each $h=t^{ip^m}r^js^k\in L_m$, we have from (6)

$$\chi(h)=\lambda(h)\sum_{u=0}^{p^m-1}(\lambda(s)^j)^u.$$

Since $\lambda(s)$ is a primitive p^m-th root of unity, it follows that

$$\sum_{u=0}^{p^m-1}(\lambda(s)^j)^u=\begin{cases}0, & (j\not\equiv 0 \bmod p^m)\\ p^m, & (j\equiv 0 \bmod p^m),\end{cases}$$

hence

$$\chi(h)=\begin{cases}0, & (h\notin T_m)\\ p^m\lambda(h), & (h\in T_m).\end{cases}$$

If we define a linear character $\bar{\mu}$ of $\bar{T}_m$ by putting $\bar{\mu}(\bar{h})=\lambda(h)$, then $\bar{\mu}$ is faithful on $\langle\bar{s}\rangle$, and

$$\chi(h)=\begin{cases}0, & (h\notin T_m)\\ p^m\bar{\mu}(\bar{h}), & (h\in T_m).\end{cases}$$

Conversely, let χ be a class function defined by (7). Since $D(L_m)=\langle s^{p^m}\rangle\subset T_m$, the linear character μ of T_m given by $\mu(h)=\bar{\mu}(\bar{h})$ is the restriction of a linear character λ of L_m. Since $\bar{\mu}$ is faithful on $\langle\bar{s}\rangle$, $\lambda(s)$ is a primitive p^m-th root of unity. It follows that λ^H is irreducible by Lemma 2 and is equal to χ, which proves Theorem 2.

We describe the structure of $\bar{T}_m$ of each group in Theorem 1. Putting $t_0=t^{p^m}$, $r_0=r^{p^m}$, we have

$$\bar{T}_m=\langle\bar{t}_0,\bar{r}_0,\bar{s}\rangle.$$

Case 1. If p is odd or if $p=2$, $n\geq 2$, then in H_1,

$$\bar{T}_m=\langle\bar{t}_0\rangle\times\langle\bar{r}_0\rangle\times\langle\bar{s}\rangle,$$
$$\langle\bar{t}_0\rangle,\langle\bar{r}_0\rangle\cong Z_{p^{n-m}},\qquad \langle\bar{s}\rangle\cong Z_{p^m},$$

in H_2,

$$\bar{T}_m=\langle\bar{t}_0\rangle\times\langle\bar{r}_0\rangle,\qquad \bar{s}=\bar{r}_0^{p^{n-m}},$$
$$\langle\bar{t}_0\rangle\cong Z_{p^{n-m}},\qquad \langle\bar{r}_0\rangle\cong Z_{p^n},$$

Case 2. If $p=2$, $n=1$, then in H_1 and H_3,

$$\bar{T}_1=\langle\bar{s}\rangle,\qquad \langle\bar{s}\rangle\cong Z_2.$$

Corollary 1. *The number of the irreducible characters of degree p^m of a representation group H of $Z_{p^n}\times Z_{p^n}$ is*

$$p^{2n-1}(p-1)p^{-m}.$$

Proof. By Theorem 2, this number is equal to the number of the linear characters of $\bar{T}_m$ which are faithful on $\langle \bar{s} \rangle$. Noting that there exists a cyclic direct factor of $\bar{T}_m$ which contains $\langle \bar{s} \rangle$, we can prove that this number is equal to $p^{2n-1}(p-1)p^{-m}$.

Now let H, H' be the two non-isomorphic representation groups of $Z_{p^n} \times Z_{p^n}$. We define a (set-theoretical) one-to-one onto mapping from $H=\langle t, r, s \rangle$ to $H'=\langle t', r', s' \rangle$: If $h \in H$ is written as $h=t^i r^j s^k$,

$$h \mapsto h' = t'^i r'^j s'^k$$

where $0 \leqq i, j, k \leqq p^n - 1$. By this mapping, a conjugacy class C of H corresponds to a conjugacy class C' of H' (see Proposition 1), thus we have a one-to-one onto correspondence between the set of conjugacy classes of H and the set of conjugacy classes of H'. Furthermore, the subgroup $T_m = \langle t^{p^m}, r^{p^m}, s \rangle$ of H corresponds to the subgroup $T'_m = \langle t'^{p^m}, r'^{p^m}, s' \rangle$ of H'.

We say that the group H and H' have the same character table if there exists a one-to-one onto mapping from the set $\{\chi\}$ of irreducible characters of H to the set $\{\chi'\}$ of irreducible characters of H' which satisfies the condition $\chi'(C') = \chi(C)$ for any conjugacy class C and any irreducible character χ.

Corollary 2. *The two non-isomorphic representation groups of $Z_{p^n} \times Z_{p^n}$ have the same character table if and only if $n=1$.*

Proof. The mapping $C \mapsto C'$ induces the one-to-one onto mapping from $\bar{T}_m$ to $\bar{T}'_m$ such that $\bar{t}_0, \bar{r}_0, \bar{s}$ corresponds to $\overline{t'_0}, \overline{r'_0}, \overline{s'}$ respectively, for each m ($m=1, 2, \cdots, n$). Let $\mathfrak{M}_m$ (resp. $\mathfrak{M}'_m$) be the set of linear characters of $\bar{T}_m$ (resp. $\overline{T'_m}$) which are faithful on $\langle \bar{s} \rangle$ (resp. $\langle \bar{s}' \rangle$). By Theorem 2, the groups H, H' have the same character table in the above sense if and only if for each m there exists a one-to-one onto mapping $\mu \mapsto \mu'$ from $\mathfrak{M}_m$ to $\mathfrak{M}'_m$ such that $\mu'(\overline{h'}) = \mu(\bar{h})$ for any $\bar{h} \in \bar{T}_m$ and any $\mu \in M_m$. It is easily seen that this happens only when $n=1$ and the corollary is proved.

Remark. The same argument can be applied to the case of central extensions of $Z_{p^l} \times Z_{p^m}$ by a cyclic group Z_{p^n} contained in the commutator subgroup. ($n \leqq m \leqq l$).

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