63. *On Surfaces of Class VII_0 with Curves*

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§1. Let S be a surface, i.e., a compact complex manifold of complex dimension 2. We write $b_i(S)$ for the i-th Betti number of S. For a divisor D on S, we write D^2 for its self intersection number. A surface S is said to be *of Class VII_0* if S is minimal and $b_1(S)=1$. When a surface S is of Class VII_0, it is well known that any divisor D on S has $D^2\leqq 0$.

In this note, we shall state theorems on a surface of Class VII_0 which has a divisor D with $D^2=0$. For this purpose, we shall construct surfaces $S_{n,\alpha,t}$ $(n>0,\ 0<|\alpha|<1,\ t\in \boldsymbol{C}^n)$, which satisfy the following conditions:

(1.1) $S_{n,\alpha,t}$ is of Class VII_0,

(1.2) $b_2(S_{n,\alpha,t})=n$,

(1.3) $S_{n,\alpha,t}$ has a connected curve $D_{n,\alpha,t}$ with $D^2_{n,\alpha,t}=0$.

Our main result is the following

Theorem 1. *Let S be a surface of Class VII_0 with $b_2(S)=n>0$. If S has a divisor $D\neq 0$ with $D^2=0$, then S is biholomorphic to $S_{n,\alpha,t}$ for some $0<|\alpha|<1$, $t\in \boldsymbol{C}^n$ and $D=mD_{n,\alpha,t}$ for some integer $m\neq 0$.*

In view of the classification theory of Kodaira on surfaces, Theorem 1 implies

Theorem 2. *Let S be a surface and C be a curve on S. Assume that*

i) *there is a non-constant holomorphic function on $S-C$,*

ii) *the number of compact irreducible curves on $S-C$ is finite. Then $S-C$ has a structure of a quasi-projective variety.*

To state theorems on deformations of $S_{n,\alpha,t}$, set

$$T_n=\{\alpha\in \boldsymbol{C}\,|\,0<|\alpha|<1\}\times \boldsymbol{C}^n,$$

$$\mathcal{S}_n=\bigcup_{(\alpha,t)\in T_n} S_{n,\alpha,t} \qquad \text{(disjoint union)},$$

$$\mathcal{D}_n=\bigcup_{(\alpha,t)\in T_n} D_{n,\alpha,t} \qquad \text{(disjoint union)},$$

$$\mathcal{A}_n=\mathcal{S}_n-\mathcal{D}_n, \qquad A_{n,\alpha,t}=S_{n,\alpha,t}-D_{n,\alpha,t}.$$

Let $\pi:\mathcal{S}_n\to T_n$ be the projection so that $\pi^{-1}(\alpha,t)=S_{n,\alpha,t}$. Let $\iota: S_{n,\alpha,t}\to\mathcal{S}_n$ be the natural inclusion. Then $\mathcal{S}_n$ has a complex structure such that the projection π is a holomorphic map of maximal rank and the inclusion ι is biholomorphic. Let Θ be the sheaf of germs of holomorphic vector fields on $S_{n,\alpha,t}$, i.e., the sheaf of germs of infinitesimal holomorphic automorphisms of $S_{n,\alpha,t}$.

Theorem 3. 1) *We have*

$$\dim H^1(S_{n,\alpha,t},\Theta)=\begin{cases}2n+1 & \text{if } t=0,\\ 2n & \text{otherwise.}\end{cases}$$

2) *$\mathcal{S}_n$ is a (complex analytic) family of surfaces $S_{n,\alpha,t}$ with the parameter space T_n. This family is not complete at any point of T_n.*

Next we consider logarithmic deformations of $S_{n,\alpha,t}$ ([4, Definition 3]). Let $\Theta(\log D_{n,\alpha,t})$ be the sheaf of germs of infinitesimal holomorphic automorphisms of $S_{n,\alpha,t}$ which send $D_{n,\alpha,t}$ into itself (cf. [4, Definition 4]). Then $H^1(S_{n,\alpha,t},\Theta(\log D_{n,\alpha,t}))$ is the space of infinitesimal logarithmic deformations.

Theorem 4. 1) *We have*

$$\dim H^1(S_{n,\alpha,t},\Theta(\log D_{n,\alpha,t}))=\begin{cases}n+1 & \text{if } t=0,\\ n & \text{otherwise.}\end{cases}$$

2) *The 7-tuple $(\mathcal{A}_n,\mathcal{S}_n,\mathcal{D}_n,\pi,T_n,(\alpha,t),\iota)$ is a family of logarithmic deformations of the triple $(A_{n,\alpha,t},S_{n,\alpha,t},D_{n,\alpha,t})$ with the parameter space T_n. This family is complete as a family of logarithmic deformations.*

§ 2. The surfaces $S_{n,\alpha,t}$ are defined as follows. Let $\boldsymbol{P}^1$ be the projective line with the homogeneous coordinates $[z_0:z_1]$. Set $W_0=\boldsymbol{P}^1\times\boldsymbol{C}$, $\Gamma_\infty=\{[0:1]\}\times\boldsymbol{C}$, $C_0=\boldsymbol{P}^1\times\{0\}$ and $p_{-1}=\Gamma_\infty\cap C_0$. We fix $n\in N$ $(n\geqq 1)$, $\alpha\in\boldsymbol{C}$ $(0<|\alpha|<1)$ and $t=(t_0,\cdots,t_{n-1})\in\boldsymbol{C}^n$. Define a birational automorphism $g_{n,\alpha,t}$ of W_0 by

$$\text{(2.1)}\quad g_{n,\alpha,t}:([z_0:z_1],w)\longmapsto\left(\left[z_0:w^nz_1+\sum_{k=0}^{n-1}t_kw^kz_0\right],\alpha w\right).$$

Note that the inverse $g^{-1}_{n,\alpha,t}$ of $g_{n,\alpha,t}$ is given by

$$g^{-1}_{n,\alpha,t}([z_0:z_1],w)=\left(\left[\alpha^{-n}w^nz_0:z_1-\sum_{k=0}^{n-1}t_k\alpha^{-k}w^kz_0\right],\alpha^{-1}w\right).$$

Then the indeterminacy set of $g_{n,\alpha,t}$ (resp. $g^{-1}_{n,\alpha,t}$) consists of one point p_{-1} (resp. $p_0=([1:t_0],0)$). We blow up W_0 at p_0 and p_{-1}:

$$W_0\xleftarrow{\sigma_1}W_1=Q_{p_{-1}}Q_{p_0}(W_0),$$

where Q_{p_0} (resp. $Q_{p_{-1}}$) denotes the quadric transformation with the center p_0 (resp. $Q_{p_0}(p_{-1})$). We set $C_1=\sigma_1^{-1}(p_0)$, $C_{-1}=\sigma_1^{-1}(p_{-1})$. We denote the proper transforms of C_0,Γ_∞ and birational automorphisms of W_1 induced from $g_{n,\alpha,t}$, $g^{-1}_{n,\alpha,t}$ by the same symbols $C_0,\Gamma_\infty,g_{n,\alpha,t}$ and $g^{-1}_{n,\alpha,t}$ respectively. Set $p_{-2}=\Gamma_\infty\cap C_{-1}$. Then the indeterminacy set of $g_{n,\alpha,t}$ in W_1 is $\{p_{-2}\}$ and that of $g^{-1}_{n,\alpha,t}$ consists of one point $p_1\in C_1$ which is different from p_0. Again we blow up W_1 at p_1 and p_{-2}. Repeating this process, we obtain a sequence of blowing-ups:

$$W_0\xleftarrow{\sigma_1}W_1\xleftarrow{\sigma_2}W_2\xleftarrow{\sigma_3}W_3\longleftarrow\cdots.$$

We regard $W_k-p_k-\Gamma_\infty$ as an open submanifold of $W_{k+1}-p_{k+1}-\Gamma_\infty$ by σ^{-1}_{k+1}. Define a non-compact surface W_∞ as the direct limit of $W_k-p_k-\Gamma_\infty$:

$$W_\infty = \underset{\overrightarrow{k}}{\lim} (W_k - p_k - \Gamma_\infty).$$

Then we have infinitely many non-singular rational curves C_j ($j \in \boldsymbol{Z}$) on W_∞ which satisfy the following conditions:

(2.2) C_j and C_{j+1} intersect transversally at p_j,

(2.3) $C_i \cap C_j = \phi$ if $i - j \neq \pm 1$.

$g_{n,\alpha,t}$ induces a holomorphic automorphism $\tilde{g}_{n,\alpha,t}$ of W_∞ such that

(2.4) $\tilde{g}_{n,\alpha,t}(C_j) = C_{j+n}$ for $j \in \boldsymbol{Z}$.

By (2.1) and (2.4), $\tilde{g}_{n,\alpha,t}$ generates a properly discontinuous group $\langle \tilde{g}_{n,\alpha,t} \rangle$ of holomorphic automorphisms of W_∞ free from fixed points. Now we define $S_{n,\alpha,t}$ to be the quotient surface of W_∞ by $\langle \tilde{g}_{n,\alpha,t} \rangle$:

$$S_{n,\alpha,t} = W_\infty / \langle \tilde{g}_{n,\alpha,t} \rangle.$$

Writing f for the canonical projection of W_∞ onto $S_{n,\alpha,t}$, set

$$D_{n,\alpha,t} = \bigcup_{i=0}^{n-1} f(C_i).$$

Then we can show that $S_{n,\alpha,t}$ and $D_{n,\alpha,t}$ satisfy the conditions (1.1)–(1.3). Moreover $A_{n,\alpha,t}$ is an affine $\boldsymbol{C}$-bundle of degree $-n$ over an elliptic curve $\boldsymbol{C}^*/\langle \alpha \rangle$, where $\langle \alpha \rangle$ is the multiplicative group generated by α.

Theorem 5. *Let A be an affine $\boldsymbol{C}$-bundle of degree $-n<0$ over the elliptic curve $\boldsymbol{C}^*/\langle \alpha \rangle$ $(0<|\alpha|<1)$. Then A is equivalent to $A_{n,\alpha,t}$ as an affine $\boldsymbol{C}$-bundle for some $t \in \boldsymbol{C}^n$.*

Theorem 6. *$S_{n,\alpha,s}$ is biholomorphic to $S_{n,\beta,t}$ $(t=(t_0, \cdots, t_{n-1}))$ if and only if $\alpha=\beta$ and there are $k \in \boldsymbol{Z}$ $(0 \leqq k < n)$, $\lambda, \kappa \in \boldsymbol{C}^*$ satisfying*

$$\kappa s = (\lambda^k t_k, \cdots, \lambda^{n-1} t_{n-1}, \alpha^k t_0, \cdots, \lambda^i \alpha^{k-i} t_i, \cdots, \lambda^{k-1} \alpha t_{k-1}), \qquad \alpha^k = \lambda^n.$$

Remarks. 1) The above construction of $S_{n,\alpha,t}$ is a generalization of that of $S_{1,\alpha,0}$ in [2, p. 57]. See also [2, Remark 4].

2) We can see that $S_{1,\alpha,0}$ is biholomorphic to the surface constructed by M. Inoue in [1] and $S_{1,\alpha,t}$ is biholomorphic to the surface constructed by Ma. Kato in [2, p. 59–60].

§ 3. Theorem 1 follows from the following Theorem 1′.

Theorem 1′. *Let S be a surface of Class VII_0 with no non-constant meromorphic function. If S has a divisor $D \neq 0$ with $D^2=0$, then S is either a Hopf surface or a surface $S_{n,\alpha,t}$.*

In fact, if S is of Class VII_0 with $b_2(S)>0$, then S is not a Hopf surface and S has no meromorphic function except constants.

To sketch a proof of Theorem 1′, let S and D be as in Theorem 1′. Let C denote the support of D. Our proof of Theorem 1′ is divided into three steps.

Step 1. We start with the following two propositions.

Proposition 1. *S is a Hopf surface if C is disconnected or C is non-singular.*

In the rest of this section, we assume that C is connected and has n singular points $(n \geqq 1)$.

Proposition 2. *There are an unramified covering $f:\tilde{S}\to S$ and a holomorphic function w on $\tilde{S}$ so that*

i) *$f^{-1}(C)$ consists of infinitely many non-singular rational curves C_j ($j\in \boldsymbol{Z}$) satisfying the conditions* (2.2), (2.3) *and*

(3.1) *the divisor (w) is $\sum_{j\in \boldsymbol{Z}} C_j$.*

ii) *the covering transformation group of $\tilde{S}$ with respect to S is generated by an element g such that*

(3.2) $g^*w=\alpha w$ $(0<|\alpha|<1)$,

(3.3) $g(C_j)=C_{j+n}$ *for* $j\in \boldsymbol{Z}$.

By (3.1) and (3.2), w induces a surjective holomorphic map

$$\psi: S-C\to \Delta=\boldsymbol{C}^*/\langle\alpha\rangle.$$

Step 2. Next we construct a compact surface $\hat{S}$ satisfying the following conditions:

i) $\hat{S}$ contains $S-C$ as an open submanifold and $\hat{C}=\hat{S}-(S-C)$ is a curve on $\hat{S}$,

ii) ψ extends to a holomorphic map $\hat{\psi}$ from $\hat{S}$ onto Δ and $\hat{\psi}$ maps $\hat{C}$ biholomorphically onto Δ.

Step 3. Using the classification theory of Kodaira on compact surfaces, we see that $\hat{\psi}:\hat{S}\to\Delta$ is a $\boldsymbol{P}^1$-bundle. Hence $S-C$ is an affine $\boldsymbol{C}$-bundle over Δ. Using conditions (3.2) and (3.3), we can show

Lemma. *There are a holomorphic function z on $\tilde{S}-\bigcup_{j\neq 0} C_j$ and $t=(t_0,\cdots,t_{n-1})\in \boldsymbol{C}^n$ such that (z,w) maps $\tilde{S}-\bigcup_{j\neq 0}C_j$ biholomorphically onto $\boldsymbol{C}^2-\{(t_0,0)\}$ and*

$$g(z,w)=\left(w^n z+\sum_{k=0}^{n-1} t_k w^k,\ \alpha w\right) \qquad \textit{for } w\neq 0.$$

From this lemma, we can prove that S is biholomorphic to $S_{n,\alpha,t}$.

To show Step 2, set $C^-=\bigcup_{j\leq 0} C_j$. We need

Proposition 3. *There is a holomorphic function ξ on a neighbourhood B of C^- in $\tilde{S}$ such that (w,z) maps $B-C^-$ biholomorphically onto $U\times U^*$ where U is a small disk containing the origin and $U^*=U-\{0\}$.*

To define the above function ξ, we construct a holomorphic 2-form φ on B. We first construct φ as a collection of formal power series in w using local coordinate systems induced from those of $f(B)$ by f. Next we prove that these power series converge absolutely and uniformly for $|w|<\varepsilon$ ($\varepsilon>0$ is sufficiently small). The proof is similar to [5] with extra arguments for the convergence of the above power series because of the non-compactness of C^-.

Set $A_u=w^{-1}(u)\cap(B-C^-)$. Then φ determines a holomorphic 1-form η_u on A_u for each $u\in U$ by the formula:

$$\varphi=\eta_u\wedge dw \qquad \text{on } A_u.$$

Let $c(u)$ be a non-vanishing holomorphic function on U. Set

$$\xi(x)=\exp\int_{a(u)}^{x} c(u)\eta_u \qquad \text{for } w(x)=u,$$

where $a(u)$ depends on u holomorphically satisfying $w(a(u))=u$. Then ξ is a single valued holomorphic function on $B-C^-$ provided that $c(u)$ is chosen properly. Moreover from the very way we construct φ, we have a nice estimate for φ so that we can show that ξ maps A_u biholomorphically onto a punctured disk. Thus we obtain Proposition 3.

Now $\hat{S}$ in Step 2 is defined as follows. Identifying $x \in w^{-1}(U^*) \cap B$ with $(w(x), \xi(x)) \in U^* \times U$, we form the union $\hat{W}$:

$$\hat{W} = w^{-1}(U^*) \cup (U^* \times U).$$

Then g extends to a holomorphic map $\hat{g}$ which maps $\hat{W}$ biholomorphically into itself. w extends to a holomorphic function $\hat{w}$ on $\hat{W}$. $\hat{S}$ is obtained from $\hat{W}$ by identifying $y \in \hat{W}$ with $\hat{g}(y)$. $\hat{\psi}$ is induced from $\hat{w}$.

Dr. M. Inoue kindly informed the author that there is a proof of Proposition 1 which is simpler than his original one. See [3].

References

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