62. *Deformation of Linear Ordinary Differential Equations. III*

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In this note we consider the integrability condition

$$d\Omega(x)-\Omega(x)^2=0 \tag{1}$$

for a system of linear total differential equations

$$dY(x)=\Omega(x)Y(x), \tag{2}$$

where $\Omega(x)$ is an $m\times m$ matrix of 1 forms of certain parameters, depending rationally on x.

In the preceding notes [1], [2], we considered (2) as the isomonodromic deformation equation for the following system

$$\frac{d}{dx}Y(x)=A(x)Y(x), \tag{3}$$

where $A(x)$ is an $m\times m$ matrix which is rational in x. Here we start from (1) and regard (3) as additional constraints which determine special solutions to the system of non linear equations (1).

The notions of τ functions, the Schlesinger transformations and the characteristic matrices introduced in [1], [2] will be generalized to the systems (1), (2) without referring to the further constraints (3). Then we shall show that the systems (1), (2) can be rewritten as bilinear differential equations in the sense of Hirota [3] by introducing the τ function and its Schlesinger transforms as dependent variables.

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1. Let us consider a formal Laurent series $\Omega(x)$ of $m\times m$ matrices of one forms

$$\Omega(x)=\sum_{j=-r}^{\infty}\Omega_j(x)\qquad \Omega_j(x)=\left(H_j\frac{da}{z}+\sum_{l=1}^{M}H_{jl}dt_l\right)z^j, \tag{4}$$

where $z=x-a$, and H_j, H_{jl} are $m\times m$ matrices of holomorphic functions of parameters $t_1,\cdots,t_m$ and a. We denote by d the exterior differentiation with respect to $t_1,\cdots,t_m$ and a.

Theorem 1. *Assume that $\Omega(x)$ is integrable,*

$$d\Omega(x)-\Omega(x)^2=0. \tag{5}$$

Moreover we assume that $\Omega_{-r}(x)$ is diagonal with mutually distinct entries:

$$\Omega_{-r}(x)_{\alpha\alpha}-\Omega_{-r}(x)_{\beta\beta}\neq 0\qquad \textit{for } \alpha\neq\beta. \tag{6}$$

Then the system

(7) $$dY(x)=\Omega(x)Y(x)$$
has a formal solution of the form
(8) $$Y(x)=\hat{Y}(x)e^{T(x)+K},$$
where
(9) $$\hat{Y}(x)=F(x)D(x),$$
$$F(x)=\sum_{j=0}^{\infty}F_jz^j,\quad F_0=1, F_j \quad \textit{is diagonal free } (j\geq 1),$$
$$D(x)=\sum_{j=0}^{\infty}D_jz^j,\quad D_0=1, D_j \quad \textit{is diagonal } (j\geq 1),$$
and
(10) $$T(x)=\sum_{j=1}^{r}T_{-j}\frac{z^{-j}}{(-j)}+T_0\log z$$
is diagonal with $dT_0=0$, and K is diagonal.

If we set
(11) $$\Lambda(x)=dT(x)+dK+d\log D(x),$$
the formal equation (7) is equivalent to
(12) $$dF(x)=\Omega(x)F(x)-F(x)\Lambda(x).$$
The series $F(x)$ and $\Lambda(x)$ are uniquely determined by (12).

By setting $z=1/x$ and dropping the da terms in $\Omega(x)$, Theorem 1 is equally valid for formal series at $x=\infty$.

Now let us start from the following data:
(13) $$T^{(\nu)}(x)=\sum_{j=1}^{r_\nu}T^{(\nu)}_{-j}\frac{z_\nu^{-j}}{(-j)}+T^{(\nu)}_0\log z_\nu,$$
$$F^{(\nu)}_0=1,\ F^{(\nu)}_1,\ \cdots,F^{(\nu)}_{r_\nu},\ G^{(\nu)}\qquad (\nu=\infty,1,\cdots,n)$$
with $G^{(\infty)}=1$. Here $z_\nu=x-a_\nu$ $(\nu\neq\infty)$, $=1/x$ $(\nu=\infty)$.
$$T^{(\nu)}_{-j}=(t^{(\nu)}_{-j,\alpha}\delta_{\alpha\beta})_{\alpha,\beta=1,\cdots,m}$$
is a diagonal matrix and $F^{(\nu)}_j$ (resp. $G^{(\nu)}$) is a diagonal free matrix (resp. an invertible matrix) of holomorphic functions of parameters a_μ $(\mu=1,\cdots,n)$ and $t^{(\mu)}_{-k,\alpha}$ $(\mu=\infty,1,\cdots,n;\ k=1,\cdots,r_\mu;\ \alpha=1,\cdots,m)$.

In accordance with [1], we set
(14) $$\sum_{j=-r_\nu}^{0}\Omega^{(\nu)}_j(x)\underset{1}{\equiv}\Big(\sum_{j=0}^{r_\nu}F^{(\nu)}_jz_\nu^j\Big)d'T^{(\nu)}(x)\Big(\sum_{j=0}^{r_\nu}F^{(\nu)}_jz_\nu^j\Big)^{-1},$$
$$\Omega^{(\nu)}_j(x)=\Big(H^{(\nu)}_j\frac{da_\nu}{z_\nu}+\sum_{k=1}^{r_\nu}\sum_{\alpha=1}^{m}H^{(\nu)}_{j,-k,\alpha}dt^{(\nu)}_{-k,\alpha}\Big)z_\nu^j,$$
$$\Omega^{(\nu)}_0=\Omega^{(\nu)}_0(x)-H^{(\nu)}_0\frac{da_\nu}{z_\nu}.$$
Here $A\underset{k}{\equiv}B$ means $A\equiv B \bmod (z_\nu^{k-1}da_\nu,\ z_\nu^k)$. We set also
(15) $$\Omega^{(\nu)}(x)=\sum_{\mu=\infty,1,\cdots,n}\ \sum_{j=-r_\nu}^{0}G^{(\nu)-1}G^{(\mu)}\Omega^{(\mu)}_j(x)G^{(\mu)-1}G^{(\nu)}$$
$$-\sum_{\mu(\neq\nu)}G^{(\nu)-1}G^{(\mu)}\Omega^{(\nu)}_0G^{(\mu)-1}G^{(\nu)},$$
$$\Theta^{(\nu)}=\Omega^{(\infty)}_0-G^{(\nu)}\Omega^{(\nu)}_0G^{(\nu)-1}.$$

Consider the following infinite system of differential equations for formal series $\hat{Y}^{(\nu)}(x)$.

$$(16) \qquad d\hat{Y}^{(\nu)}(x)=\Omega^{(\nu)}(x)\hat{Y}^{(\nu)}(x)-\hat{Y}^{(\nu)}(x)d'T^{(\nu)}(x), \qquad dG^{(\nu)}=\Theta^{(\nu)}G^{(\nu)}.$$

Theorem 1 asserts that if $F_j^{(\nu)}$ and $G^{(\nu)}$ ($\nu=\infty, 1, \cdots, n; j=1, \cdots, r_\nu$) satisfy the followihg systems

$$(17) \qquad \begin{aligned} &d\Omega^{(\nu)}(x)-\Omega^{(\nu)}(x)^2=0 && (\nu=\infty, 1, \cdots, n), \\ &dG^{(\nu)}=\Theta^{(\nu)}G^{(\nu)} && (\nu=1, \cdots, n), \end{aligned}$$

then we can find $\hat{Y}^{(\nu)}(x)$ so that (16) is valid. Note that by the uniqueness of $\Lambda(x)$ in (12), $dK=0$ in the present case. For $\nu\neq\infty$, $\hat{Y}^{(\nu)}(x)$ is unique. For $\nu=\infty$, or also for $\nu\neq\infty$ if we do not consider the dependence on a_ν, $\hat{Y}^{(\nu)}(x)$ is unique up to the following modification.

$$(18) \qquad \hat{Y}^{(\nu)}(x)\to\hat{Y}^{(\nu)}(x)\left(\sum_{j=0}^{\infty}K_j^{(\nu)}z_\nu^j\right)$$

where $K_0^{(\nu)}=1$ and $K_j^{(\nu)}$ is a *constant* diagonal matrix.

As is well-known, the system (17) for the unknown matrices $F_j^{(\nu)}$, $G^{(\nu)}$ ($\nu=\infty, 1, \cdots, n; j=1, \cdots, r_\nu$) contains a wide class of completely integrable non linear differential equations [4], [5]. We can define the τ functions and the Schlesinger transformations for solutions to the system (17) through the matrices $\hat{Y}^{(\nu)}(x)$ ($\nu=\infty, 1, \cdots, n$) satisfying (16) as in [1], [2]. Note that the undetermined factor (18) causes only a linear difference in $d\log\tau$.

2. We introduce the characteristic matrices $G^{(\mu,\nu)(k,l)}$ through the same equations as (9)–(12) of [1]. We show that the matrix elements $\tau G_{\alpha\beta}^{(\mu,\nu)(k,l)}$ are expressed in terms of the derivatives of the τ function and its Schlesinger transformations.

Proposition 2. *The following recursion formulas hold for* $\mu, \nu=1, \cdots, n, \infty$.

$$(19) \qquad \tau G_{\alpha\beta}^{(\mu,\nu)(k,l+1)}=\delta_{\mu\nu}\delta_{\alpha\beta}\frac{1}{l}\sum_{s=1}^{l}\tau G_{\alpha\alpha}^{(\nu,\nu)(k+s,l-s+1)} + \frac{1}{l}\sum_{s=1}^{l}s\frac{\partial}{\partial t_{-s\beta}^{(\nu)}}(\tau G_{\alpha\beta}^{(\mu,\nu)(k,l-s+1)}) \qquad (1\leq k,\ 1\leq l\leq r_\nu)$$

$$(20) \qquad \tau G_{\alpha\beta}^{(\mu,\nu)(k+1,l)}=\delta_{\mu\nu}\delta_{\alpha\beta}\frac{1}{k}\sum_{s=1}^{k}\tau G_{\alpha\alpha}^{(\nu,\nu)(k-s+1,l+s)} - \frac{1}{k}\sum_{s=1}^{k}s\frac{\partial}{\partial t_{-s\alpha}^{(\mu)}}(\tau G_{\alpha\beta}^{(\mu,\nu)(k-s+1,l)}) \qquad (1\leq k\leq r_\mu,\ 1\leq l).$$

Using (19), (20) we may express $\tau G_{\alpha\beta}^{(\mu,\nu)(k,l)}$ in terms of τ and $\tau G_{\alpha\beta}^{(\mu,\nu)(1,1)}=\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}$, the latter being the τ function corresponding to an elementary Schlesinger transformation of type $\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}$ ([2]) with $(\mu, \alpha)\neq(\nu, \beta)$. For this purpose we introduce polynomials $p_k(x)=p_k(x_1, \cdots, x_k)$ through the formal series

$$(21) \qquad \exp\left(\sum_{j=1}^{\infty}\varepsilon^j x_j\right)=\sum_{k=0}^{\infty}\varepsilon^k p_k(x).$$

Explicitly they are given by

$$p_1(x)=x_1,\quad p_2(x)=x_2+\frac{1}{2}x_1^2,\quad p_3(x)=x_3+x_2x_1+\frac{1}{6}x_1^3, \tag{22}$$
$$p_4(x)=x_4+x_3x_1+\frac{1}{2}x_2^2+\frac{1}{2}x_2x_1^2+\frac{1}{24}x_1^4,\ \cdots$$
$$p_k(x)=\sum_{s=1}^{k}\ \sum_{\substack{l_1+2l_2+\cdots+sl_s=k\\ l_1,\cdots,l_s\geq 1}}\frac{1}{l_1!\cdots l_s!}x_1^{l_1}\cdots x_s^{l_s}.$$

We set

$$p_k(\partial_\alpha^{(\mu)})=p_k\left(\frac{\partial}{\partial t_{-1\alpha}^{(\mu)}},\cdots,\frac{\partial}{\partial t_{-k\alpha}^{(\mu)}}\right)\qquad\text{for } 1\leq k\leq r_\mu.$$

We have then

Theorem 3.

$(\mu,\alpha)\neq(\nu,\beta)$

$$\tau G_{\alpha\beta}^{(\mu,\nu)(k,l)}=p_{k-1}(-\partial_\alpha^{(\mu)})p_{l-1}(\partial_\beta^{(\nu)})\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix} \tag{23}$$
$$(1\leq k\leq r_\mu+1,\ 1\leq l\leq r_\nu+1).$$

$(\mu,\alpha)=(\nu,\beta)$

$$\tau G_{\alpha\alpha}^{(\mu,\mu)(k,l)}=p_{kl}(\partial_\alpha^{(\mu)})\tau\qquad(1\leq k,\ l\leq r_\mu) \tag{24}$$
$$p_{kl}(\partial_\alpha^{(\mu)})=\sum_{j=1}^{k}p_{k-j}(-\partial_\alpha^{(\mu)})\cdot p_{l+j-1}(\partial_\alpha^{(\mu)})$$
$$=-\sum_{j=1}^{l}p_{k+j-1}(-\partial_\alpha^{(\mu)})\cdot p_{l-j}(\partial_\alpha^{(\mu)}).$$

The first few terms are given in the following table:

(25)

	$l=1$	$l=2$	$l=3$
$k=1$	$\tau_{\alpha\beta}$	$\dfrac{\partial\tau_{\alpha\beta}}{\partial t_{-1\beta}^{(\nu)}}$	$\left(\dfrac{\partial}{\partial t_{-2\beta}^{(\nu)}}+\dfrac{1}{2}\dfrac{\partial^2}{\partial t_{-1\beta}^{(\nu)2}}\right)\tau_{\alpha\beta}$
$k=2$	$-\dfrac{\partial\tau_{\alpha\beta}}{\partial t_{-1\alpha}^{(\mu)}}$	$-\dfrac{\partial^2\tau_{\alpha\beta}}{\partial t_{-1\alpha}^{(\mu)}\partial t_{-1\beta}^{(\nu)}}$	$-\dfrac{\partial}{\partial t_{-1\alpha}^{(\mu)}}\left(\dfrac{\partial}{\partial t_{-2\beta}^{(\nu)}}+\dfrac{1}{2}\dfrac{\partial^2}{\partial t_{-1\beta}^{(\nu)2}}\right)\tau_{\alpha\beta}$

$$\left(\tau_{\alpha\beta}=\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}\right)$$

(26)

	$l=1$	$l=2$
$k=1$	$\dfrac{\partial\tau}{\partial t_{-1\alpha}^{(\mu)}}$	$\left(\dfrac{\partial}{\partial t_{-2\alpha}^{(\mu)}}+\dfrac{1}{2}\dfrac{\partial^2}{\partial t_{-1\alpha}^{(\mu)2}}\right)\tau$
$k=2$	$\left(\dfrac{\partial}{\partial t_{-2\alpha}^{(\mu)}}-\dfrac{1}{2}\dfrac{\partial^2}{\partial t_{-1\alpha}^{(\mu)2}}\right)\tau$	$\left(\dfrac{\partial}{\partial t_{-3\alpha}^{(\mu)}}-\dfrac{1}{3}\dfrac{\partial^3}{\partial t_{-1\alpha}^{(\mu)3}}\right)\tau$

	$l=3$
$k=1$	$\left(\dfrac{\partial}{\partial t_{-3\alpha}^{(\mu)}}+\dfrac{\partial^2}{\partial t_{-2\alpha}^{(\mu)}\partial t_{-1\alpha}^{(\mu)}}+\dfrac{1}{6}\dfrac{\partial^3}{\partial t_{-1\alpha}^{(\mu)3}}\right)\tau$
$k=2$	$\left(\dfrac{\partial}{\partial t_{-4\alpha}^{(\mu)}}+\dfrac{1}{2}\dfrac{\partial^2}{\partial t_{-2\alpha}^{(\mu)2}}-\dfrac{1}{2}\dfrac{\partial^3}{\partial t_{-2\alpha}^{(\mu)}\partial t_{-1\alpha}^{(\mu)2}}-\dfrac{1}{8}\dfrac{\partial^4}{\partial t_{-1\alpha}^{(\mu)4}}\right)\tau.$

From equation (24) with $k=l=1$ and the definition (21) of p_k, we find in particular that

$$\tau\hat{Y}_{\alpha\alpha}^{(\mu)}(x)\equiv\tau(\cdots,t_{-1\alpha}^{(\mu)}+z_\mu,\ t_{-2\alpha}^{(\mu)}+z_\mu^2,\ \cdots,\ t_{-r_\mu\alpha}^{(\mu)}+z_\mu^{r_\mu},\ \cdots) \tag{27}$$
$$\bmod O(z_\mu^{r_\mu+1}),\qquad(\mu=1,\cdots,n,\infty;\ \alpha=1,\cdots,m)$$

where in the right hand side the variables $t^{(\mu)}_{-k\alpha}$ are shifted by z^k_μ ($k=1, \cdots, r_\mu$) and the remaining ones are fixed. These identities characterize the τ function to within a multiplicative constant.

3. From the differential equations (16) for $\hat{Y}(x)$, G and the defining equations (9)–(12) in [1] of the characteristic matrices, it is possible to derive differential equations for $G^{(\mu,\nu)(k,l)}$. We find that they are of the following simple form:

$$-s\frac{\partial}{\partial t^{(\lambda)}_{-s\gamma}}G^{(\mu,\nu)(k,l)}_{\alpha\beta}=\sum_{j\in Z}G^{(\mu,\lambda)(k,j)}_{\alpha\gamma}G^{(\lambda,\nu)(s-j+1,l)}_{\gamma\beta} \tag{28}$$

$$(\lambda,\mu,\nu=1,\cdots,n,\infty;\ \alpha,\beta,\gamma=1,\cdots,m;\ s=1,\cdots,r_\lambda;\ k,l\geq 1)$$

$$\begin{aligned}-\frac{\partial}{\partial a_\lambda}G^{(\mu,\nu)(k,l)}_{\alpha\beta}=&\sum_{s=0}^{r_\lambda}\sum_{\gamma=1}^{m}t^{(\lambda)}_{-s\gamma}\sum_{j\in Z}G^{(\mu,\lambda)(k,j+1)}_{\alpha\gamma}G^{(\lambda,\nu)(s-j+1,l)}_{\gamma\beta}\\&-\delta_{\mu\lambda}kG^{(\mu,\nu)(k+1,l)}_{\alpha\beta}-\delta_{\lambda\nu}lG^{(\mu,\nu)(k,l+1)}_{\alpha\beta}\end{aligned} \tag{29}$$

$$(\lambda\neq\infty,\ \mu,\nu=1,\cdots,n,\infty;\ \alpha,\beta=1,\cdots,m;\ k,l\geq 1).$$

These equations are originally consequences of (1) and contain (16) as special cases. The latter in turn implies the integrability condition (1). In this sense (28), (29) are equivalent to the original equations (1).

Let us write down (28), (29) in terms of the τ function and its (elementary) Schlesinger transforms $\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}$, by the aid of Theorem 3. The results are expressed as Hirota's bilinear differential equations ([3]).

$$\begin{aligned}&sD^{(\lambda)}_{-s\gamma}p_k\left(\frac{1}{2}D^{(\mu)}_\alpha\right)p_l\left(\frac{1}{2}D^{(\nu)}_\beta\right)\tau\cdot\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}\\&=p_{s-1}(D^{(\lambda)}_\gamma)p_k\left(-\frac{1}{2}D^{(\mu)}_\alpha\right)p_l\left(-\frac{1}{2}D^{(\nu)}_\beta\right)\tau\begin{Bmatrix}\mu\lambda\\ \alpha\gamma\end{Bmatrix}\cdot\tau\begin{Bmatrix}\lambda\nu\\ \gamma\beta\end{Bmatrix}\end{aligned} \tag{30}$$

$$((\mu\alpha),(\nu\beta),(\lambda\gamma):\ \text{distinct})$$

$$\begin{aligned}&\left[sD^{(\mu)}_{-s\alpha}p_k\left(\frac{1}{2}D^{(\mu)}_\alpha\right)-\sum_{i=0}^{k}p_{s+i}(D^{(\mu)}_\alpha)p_{k-i}\left(-\frac{1}{2}D^{(\mu)}_\alpha\right)\right]p_l\left(-\frac{1}{2}D^{(\nu)}_\beta\right)\\&\qquad\times\tau\cdot\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}=0\qquad((\mu,\alpha)\neq(\nu,\beta))\\&\left[sD^{(\nu)}_{-s\beta}p_l\left(-\frac{1}{2}D^{(\nu)}_\beta\right)+\sum_{i=0}^{l}p_{s+i}(-D^{(\nu)}_\beta)p_{l-i}\left(\frac{1}{2}D^{(\nu)}_\beta\right)\right]p_k\left(\frac{1}{2}D^{(\mu)}_\alpha\right)\\&\qquad\times\tau\cdot\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}=0\qquad((\mu,\alpha)\neq(\nu,\beta))\end{aligned} \tag{31}$$

$$-\frac{rs}{2}D^{(\mu)}_{-r\alpha}D^{(\nu)}_{-s\beta}\tau\cdot\tau=p_{r-1}(-D^{(\mu)}_\alpha)p_{s-1}(D^{(\nu)}_\beta)\tau\begin{Bmatrix}\mu\nu\\ \alpha\beta\end{Bmatrix}\cdot\tau\begin{Bmatrix}\nu\mu\\ \beta\alpha\end{Bmatrix} \tag{32}$$

$$((\mu,\alpha)\neq(\nu,\beta))$$

$$\left[-\frac{rs}{2}D^{(\nu)}_{-r\alpha}D^{(\nu)}_{-s\alpha}+\sum_{k=1}^{r}kp_{s+k}(D^{(\nu)}_\alpha)p_{r-k}(-D^{(\nu)}_\alpha)\right]\tau\cdot\tau=0. \tag{33}$$

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