61. *Polynomial Hamiltonians Associated with Painlevé Equations. I*[*)]

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1. Introduction. The purpose of this note is to study Hamiltonians associated with the six equations of Painlevé, and τ-functions related to Hamiltonians. The Painlevé equations are given by the following table:

$\mathrm{P_I}$ $\quad \lambda''=6\lambda^2+t$

$\mathrm{P_{II}}$ $\quad \lambda''=2\lambda^3+t\lambda+\alpha$

$\mathrm{P_{III}}$ $\quad \lambda''=\dfrac{1}{\lambda}(\lambda')^2-\dfrac{1}{t}\lambda'+\dfrac{1}{t}(\alpha\lambda^2+\beta)+\gamma\lambda^3+\dfrac{\delta}{\lambda}$

$\mathrm{P_{IV}}$ $\quad \lambda''=\dfrac{1}{2\lambda}(\lambda')^2+\dfrac{3}{2}\lambda^3+4t\lambda^2+2(t^2-\alpha)\lambda+\dfrac{\beta}{\lambda}$

$\mathrm{P_V}$ $\quad \lambda''=\left(\dfrac{1}{2\lambda}+\dfrac{1}{\lambda-1}\right)(\lambda')^2-\dfrac{1}{t}\lambda'+\dfrac{(\lambda-1)^2}{t^2}\left(\alpha\lambda+\dfrac{\beta}{\lambda}\right)+\dfrac{\gamma}{t}\lambda+\dfrac{\lambda(\lambda+1)}{\lambda-1}\delta$

$\mathrm{P_{VI}}$ $\quad \lambda''=\dfrac{1}{2}\left(\dfrac{1}{\lambda}+\dfrac{1}{\lambda-1}+\dfrac{1}{\lambda-1}\right)(\lambda')^2-\left(\dfrac{1}{t}+\dfrac{1}{t-1}+\dfrac{1}{\lambda-t}\right)\lambda'$

$$+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{\lambda^2}+\lambda\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}\right],$$

where α, β, γ and δ denote complex constants.

These equations are equivalent to the Hamiltonian systems

$$\left\{\begin{aligned}\frac{d\lambda}{dt}&=\frac{\partial \mathrm{H}}{\partial\mu}\\ \frac{d\mu}{dt}&=-\frac{\partial \mathrm{H}}{\partial\lambda}\end{aligned}\right.\tag{1}$$

with a polynomial or rational Hamiltonian $\mathrm{H}=\mathrm{H}(t\,;\lambda,\mu)$. Historically this fact was first remarked by J. Malmquist, in his paper studying polynomial systems of differential equations without movable branch points and some explicit forms of polynomial Hamiltonians were given for the Painlevé equations except for the third one ([1], p. 86). Recently, the author showed that the Painlevé equations are equivalent to systems of the form (1) and gave each system a geometric interpretation ([2], p. 47).

2. Isomonodromic deformations. Consider firstly the linear differential equation

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$$\frac{d^2y}{dx^2}+p_1(x,t)\frac{dy}{dx}+p_2(x,t)y=0, \tag{2}$$

depending on a parameter t. We suppose that (2) is of the Fuchsian type with the five singularities $x=0$, 1, t, λ and ∞. The Riemannian scheme of (2) is given by

$$\mathrm{R}_{\mathrm{VI}} \qquad \begin{pmatrix} x=0 & x=1 & x=t & x=\lambda & x=\infty \\ \theta_0' & \theta_1' & \theta_t' & \theta_\lambda' & \nu \\ \theta_0 & \theta_1 & \theta_t & \theta_\lambda'+2 & \nu+\theta_\infty \end{pmatrix},$$

The Fuchsian relation for (2) reads

$$\theta_0+\theta_0'+\theta_1+\theta_1'+\theta_t+\theta_t'+\theta_\infty+2\theta_\lambda'+2\nu=1. \tag{3}$$

Now we study the isomonodromic deformation of (2) under the following assumption: (A) *$x=\lambda$ is not a logarithmic singular point.* Viewing t as a deformation parameter, we obtain

Proposition 1. *Under the assumption* (A), *the isomonodromic deformation is governed by a Hamiltonian system of the form* (1) *with the Hamiltonian*

$$\mathrm{H}=-\operatorname*{R\acute{e}s}_{x=t} p_2(x,t), \tag{4}$$

and the conjugate canonical variable

$$\mu=\operatorname*{R\acute{e}s}_{x=\lambda} p_2(x,t). \tag{5}$$

The system thus obtained is equivalent to the equation P_{VI}.

It follows from (A) that the Hamiltonian H is written as rational function in λ and μ. Moreover, we obtain

Corollary. *If the θ_Δ' vanish for $\Delta=0$, 1, t or λ, then the Hamiltonian is a polynomial in λ and μ.*

3. Polynomial hamiltonians. The result of the Proposition 1 is valid also for the other Painlevé equations. The isomonodromic deformation of linear equation of the form (2) determines the polynomial Hamiltonian associated with each of the equations. Here we give them below:

Table (H):

H_{I} $\quad \dfrac{1}{2}\mu^2-2\lambda^3-\lambda t$

H_{II} $\quad \dfrac{1}{2}\mu^2-\left(\lambda^2+\dfrac{t}{2}\right)\mu-\left(\alpha+\dfrac{1}{2}\right)\lambda$

$\mathrm{H}_{\mathrm{III}}$ $\quad \dfrac{1}{t}[\lambda^2\mu^2-\{2\eta_\infty t\lambda^2+(2\theta_0+1)\lambda-2\eta_0 t\}\mu+2\eta_\infty(\theta_\infty+\theta_0)t\lambda]$

H_{IV} $\quad 2\lambda\mu^2-\{\lambda^2+2t\lambda+2\theta_0\}\mu+\theta_\infty\lambda$

H_{V} $\quad \dfrac{1}{t}\Big[\lambda(\lambda-1)^2\mu^2-\{\theta_0(\lambda-1)^2+\theta_1\lambda(\lambda-1)-\eta_1 t\lambda\}\mu$

$$+\frac{1}{4}\{(\theta_0+\theta_1)^2-\theta_\infty^2\}(\lambda-1)]$$

$$\mathrm{H_{VI}}\quad \frac{1}{t(t-1)}\Big[\lambda(\lambda-1)(\lambda-t)\mu^2 \\ -\{\theta_0(\lambda-1)(\lambda-t)+\theta_1\lambda(\lambda-t)+(\theta_t-1)\lambda(\lambda-1)\}\mu \\ +\frac{1}{4}\{(\theta_0+\theta_1+\theta_t-1)^2-\theta_\infty^2\}(\lambda-t)\Big].$$

Here the constants in $\mathrm{H_{III}}\cdots\mathrm{H_{VI}}$ are connected to α, β, γ, δ of the equations as follows:

$\mathrm{H_{III}}$ $\alpha=-4\eta_\infty\theta_\infty$, $\beta=4\eta_0(\theta_0+1)$, $\gamma=4\eta_\infty^2$, $\delta=-4\eta_0^2$,

$\mathrm{H_{IV}}$ $\alpha=-\theta_0+2\theta_\infty+1$, $\beta=-2\theta^2$,

$\mathrm{H_V}$ $\alpha=\frac{1}{2}\theta_\infty^2$, $\beta=-\frac{1}{2}\theta_0^2$, $\gamma=-\eta_1(\theta_1+1)$, $\delta=-\frac{1}{2}\eta_1^2$,

$\mathrm{H_{VI}}$ $\alpha=\frac{1}{2}\theta_\infty^2$, $\beta=-\frac{1}{2}\theta_0^2$, $\gamma=\frac{1}{2}\theta_1^2$, $\delta=\frac{1}{2}(1-\theta_t^2)$.

Remark 1. By taking into consideration the first equation of (1), the canonical variable μ is a rational function in λ and λ'. Thus the Hamiltonians are written as rational functions of λ and its first derivative.

4. τ-functions related to the Hamiltonians. Let $\mathrm{H}_J(t;\lambda,\mu)$ be the Hamiltonian associated with the equation $\mathrm{P}_J(J=\mathrm{I}\cdots\mathrm{IV})$ and let Ξ_J be the set of fixed critical points of P_J and set $\boldsymbol{B}_J=\boldsymbol{P}^1(\boldsymbol{C})-\Xi_J$. Then any solutions $(\lambda(t),\mu(t))$ of System (1) is meromorphic on the universal covering surface $\tilde{\boldsymbol{B}}_J$ of $\boldsymbol{B}_J$ and so is the function $\mathrm{H}_J(t;\lambda(t),\mu(t))$.

Now we state the theorem:

Theorem. *Let* H_J *be the Hamiltonian associated with* P_J, *given in the Table* (H). *Then the function*

$$\tau_J(t)=\exp\int^t \mathrm{H}_J(s;\lambda(s),\mu(s))ds \tag{6}$$

is holomorphic on $\tilde{\boldsymbol{B}}_J$, *and has only simple zeros on* $\tilde{\boldsymbol{B}}_J$.

The function $\tau_J(t)$ defined by (6) is called τ-function related to the Hamiltonian H_J. This theorem is proved by the use of Laurent expansions of $(\lambda(t),\mu(t))$ around poles.

5. Examples. In [3], the scaling function $F(t)$, obtained in the certain scaling limit of the spin-spin correlation function for the two-dimensional Ising model on a square lattice, was exactly computed. The principal part of the function $F(t)$ is given by

$$F(t)\sim\exp\int_{t'}^\infty \frac{s}{4\lambda^2}\Big[(1-\lambda(s)^2)^2-\Big(\frac{d}{ds}\lambda(s)\Big)^2\Big]ds,\qquad t'=\frac{1}{2}t, \tag{7}$$

where $\lambda=\lambda(s)$ is a solution of $\mathrm{P_{III}}$ with

$$\alpha=\beta=0,\quad \gamma=1,\quad \delta=-1.$$

Put in $\mathrm{H_{III}}$ of Table (H)

$$\eta_\infty=\eta_0=\frac{1}{2}\quad \theta_0=-1,\quad \theta_\infty=0,$$

and consider the two polynomials defined by

(8) $$\mathrm{H}(t;\lambda,\mu)=\mathrm{H}_{\mathrm{III}}+\frac{1}{4t}\qquad \tilde{\mathrm{H}}(t;\lambda,\mu)=-\mathrm{H}(-t;\lambda,\mu).$$

In this case $\mathrm{P}_{\mathrm{III}}$ remains invariant for the change of variable $t\to -t$, and hence both $\mathrm{H}(t;\lambda,\mu)$ and $\tilde{\mathrm{H}}(t;\lambda,\mu)$ are Hamiltonians. We can deduce from (6)–(8)

$$F(t)\sim\sqrt{\tau(t')\tilde{\tau}(t')},\qquad t'=\frac{1}{2}t,$$

τ and $\tilde{\tau}$ denoting the τ-functions related to H and $\tilde{\mathrm{H}}$ respectively.

As the second example we consider a one-dimensional N-body problem in a periodic box, $0\leq x\leq L$, with the Hamiltonian

$$-\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}+c\sum_{i<j}\delta(x_i-x_j).$$

In [4], for the thermodynamic limit $\rho(|x-x'|)$ of the one particle density matrix at zero temperature, the following expression was given by

(9) $$\rho(t)=\rho_0\exp\int_0^t\left[\frac{s}{4\lambda(\lambda-1)^2}\left\{\left(\frac{d\lambda}{ds}\right)^2+4\lambda^2\right\}-\frac{(\lambda+1)^2}{4s}\right]ds,$$

where $\lambda=\lambda(s)$ is a solution of P_{V} with

$$\alpha=\frac{1}{2},\quad \beta=-\frac{1}{2},\quad \gamma=-2\sqrt{-1},\quad \delta=2.$$

Now let H_{V} be the Hamiltonian in Table (H) with

$$\theta_0=\theta_\infty=1,\quad \eta_1=-2\sqrt{-1},\quad \theta_1=0,$$

and put

(10) $$\mathrm{H}(t;\lambda,\mu)=\mathrm{H}_{\mathrm{V}}(t;\lambda,\mu)+\sqrt{-1}-\frac{1}{t}.$$

Then it is not difficult to verify that the integrant of (9) coincides with (10), hence the function defined by (9) is nothing but the τ-function related to the Hamiltonian (10).

Remark 2. Consider the rational function of λ and $\bar{\mu}$,

(11) $$\mathrm{K}=\frac{1}{t}\left[\lambda^2\bar{\mu}^2-3\lambda\bar{\mu}-\frac{t^2}{4\lambda^2}-\frac{1}{4}t^2\lambda^2+\frac{9}{4}-\frac{t^2}{2}\right],$$

and the Hamiltonian system (1) with the Hamiltonian (11). This system is equivalent to $\mathrm{P}_{\mathrm{III}}$, arising in the study of the two-dimensional Ising model. Moreover, we have from (7)

$$F(t)\sim\exp\int^{t'}\mathrm{K}(s;\lambda(s),\bar{\mu}(s))ds,\qquad t'=\frac{1}{2}t.$$

Define the function $\bar{\tau}(t)$ by (6) with $\mathrm{H}=\mathrm{K}$. It is not holomorphic on $\tilde{\boldsymbol{B}}_{\mathrm{III}}$, while

$$\bar{\tau}(t)^2=\tau(t)\tilde{\tau}(t).$$

The Hamiltonian (11) is derived from the isomonodromic deformation of a equation of the form

$$\frac{d^2z}{dx^2}=p(x,t)z.$$

6. Degeneration of the Hamiltonians. Let $L_{(J)}$ be the linear equation of the form (2) such that its isomonodromic deformation induces the polynomial Hamiltonian H_J of Table (H). As is well-known, the Painlevé equation P_{VI} yields the other five ones by a process of coalescence. This fact stands also for the Hamiltonians, and the process of step-by-step degeneration is carried out according to the following scheme:

$$\begin{tikzcd} & & H_{III} \arrow[dr] & & \\ H_{VI} \arrow[r] & H_{V} \arrow[ur] \arrow[dr] & & H_{II} \arrow[r] & H_{I}. \\ & & H_{IV} \arrow[ur] & & \end{tikzcd}$$

For example, in H_{VI} replace t by $1+\varepsilon t$ and H_{VI} by $\varepsilon^{-1}H_{(\varepsilon)}$. This defines a canonical transformation with the parameter ε. Moreover, if we substitute $\eta_1\varepsilon^{-1}+\theta_1+1$ for θ_1 and $-\eta_1\varepsilon^{-1}$ for θ_t, $H_{(\varepsilon)}(t;\lambda,\mu)$ is holomorphic in ε and $H_V=H_{(0)}(t;\lambda,\mu)$. This replacement and succeeding limitation cause simultaneously the confluence of the singular point $x=t$ to $x=1$, hence the linear equation $L_{(VI)}$ degenerates to $L_{(V)}$. We have $L_{(J)}$ $(J=\mathrm{I}\cdots\mathrm{V})$ from $L_{(VI)}$ by the following process of step-by-step confluence of singularities:

$$\begin{tikzcd} & & L_{(III)} \arrow[dr] & & \\ L_{(VI)} \arrow[r] & L_{(V)} \arrow[ur] \arrow[dr] & & L_{(II)} \arrow[r] & L_{(I)}. \\ & & L_{(VI)} \arrow[ur] & & \end{tikzcd}$$

References

[1] J. Malmquist: Sur les équations différentielles du second ordre dont l'intégral général a ses points critiques fixes. Ark. Mat. Astr. Fys., **17**, 1–89 (1922–23).

[2] K. Okamoto: Sur les feuilletages associés aux équations du second ordre à points critiques fixes de P. Painlevé. Jap. J. Math., **5**, 1–79 (1979).

[3] T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch: Spin-Spin correlation functions for the two-dimensional Ising model. Exact theory in the scaling region. Phys. Rev. B, **13**, 316–374 (1976).

[4] M. Jimbo, T. Miwa, Y. Môri, and M. Sato: Density matrix of impenetrable bose gas and the fifth Painlevé transcendent. RIMS preprint, no. 303 (1979).