59. A Note on Quasilinear Evolution Equations

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(Communicated by Kôsaku YOSIDA, M. J. A., June 12, 1980)

§1. **Introduction.** In this note we give a generalization of the result of Massey [2] who proved analyticity in t of solutions to quasilinear evolution equations

$$\frac{du}{dt}+A(t,u)u=f(t,u), \qquad 0\leqq t\leqq T, \tag{1.1}$$

$$u(0)=u_0. \tag{1.2}$$

The unknown, u, is a function of t with values in a Banach space X. For fixed t and $v\in X$, the linear operator $-A(t,v)$ is the generator of an analytic semigroup in X and $f(t,v)\in X$. We consider the equation (1.1) under the assumption that the domain $D(A(t,u)^h)$ of $A(t,u)^h$ is independent of t,u for some $h>0$, while Massey discussed it in the case that $D(A(t,u))$ is constant.

In the following $L(X,Y)$ is the space of linear operators from normed space X to normed space Y, and $B(X,Y)$ is the space of bounded linear operators from normed space X to normed space Y. $L(X)=L(X,X)$ and $B(X)=B(X,X)$. $\| \ \|$ will be used for the norm in both X and $B(X)$.

The author wishes to express her hearty thanks to Prof. Y. Kōmura for his kind advices and encouragements.

§2. **The main result.** We shall make the following assumptions:

A-1°) $u_0\in D(A_0)$ and $A_0^{-\alpha}$ is a well-defined operator $\in B(X)$ where $A_0\equiv A(0,u_0)$.

A-2°) There exist $h=1/m$, where m is an integer, $m\geqq 2$, $R>0$, $T_0>0$, $\phi_0>0$ and $0\leqq\alpha<h$, such that $A(t,A_0^{-\alpha}w)$ is a well-defined operator $\in L(X)$ for each $t\in\Sigma_0\equiv\{t\in C;|\arg t|<\phi_0,\ 0\leqq|t|<T_0\}$ and $w\in N \equiv\{w\in X;\|w-A_0^{\alpha}u_0\|<R\}$.

A-3°) For any $t\in\Sigma_0$ and $w\in N$

$$\begin{cases}\text{the resolvent of } A(t,A_0^{-\alpha}w) \text{ contains the left half-plane and}\\ \text{there exists } C_1 \text{ such that } \|(\lambda-A(t,A_0^{-\alpha}w))^{-1}\|\leqq C_1(1+|\lambda|)^{-1}.\end{cases} \tag{2.1}$$

A-4°) The domain $D(A(t,A_0^{-\alpha}w)^h)=D$ of $A(t,A_0^{-\alpha}w)^h$ is independent of $t\in\Sigma_0$ and $w\in N$.

A-5°) The map $\Phi:(t,w)\mapsto A(t,A_0^{-\alpha}w)^hA_0^{-h}$ is analytic from $(\Sigma_0\backslash\{0\})\times N$ to $B(X)$.

A-6°) There exist $C_2, C_3, \sigma, 1-h<\sigma\leqq 1$ such that

$$\|A(t,A_0^{-\alpha}w)^hA(s,A_0^{-\alpha}v)^{-h}\|\leqq C_2 \qquad t,s\in\Sigma_0,\ w,v\in N, \tag{2.2}$$

(2.3) $$\|A(t, A_0^{-\alpha}w)^h A(s, A_0^{-\alpha}v)^{-h} - I\| \leqq C_3\{|t-s|^\sigma + \|w-v\|\} \qquad t, s \in \Sigma_0,\ w, v \in N.$$

A-7°) $f(t, A_0^{-\alpha}w)$ is defined and belongs to X for each $t \in \Sigma_0$ and $w \in N$, and there exists C_4 such that

(2.4) $$\|f(t, A_0^{-\alpha}w) - f(s, A_0^{-\alpha}v)\| \leqq C_4\{|t-s|^\sigma + \|w-v\|\} \qquad t, s \in \Sigma_0,\ w, v \in N.$$

A-8°) The map $\Psi : (t, w) \mapsto f(t, A_0^{-\alpha}w)$ is analytic from $(\Sigma_0 \backslash \{0\}) \times N$ into X.

These constants C_i $(i=1, 2, 3, 4)$ do not depend on t, s, w, v.

Theorem. *Let the assumptions* A-1°)–A-8°) *hold. Then there exist* $T, 0<T\leqq T_0, \phi, 0<\phi\leqq\phi_0, K>0, k, 1-h<k<1$ *and a unique continuous function* u *mapping* $\Sigma \equiv \{t \in C; |\arg t|<\phi, 0\leqq|t|<T\}$ *into* X *such that* $u(0)=u_0$, $u(t) \in D(A(t, u(t)))$ *and* $\|A_0^\alpha u(t) - A_0^\alpha u_0\| < R$ *for* $t \in \Sigma\backslash\{0\}$, $u : \Sigma\backslash\{0\} \to X$ *is analytic,* $du/dt + A(t, u(t))u(t) = f(t, u(t))$ *for* $t \in \Sigma\backslash\{0\}$, *and* $\|A_0^\alpha u(t) - A_0^\alpha u_0\| \leqq K|t|^k$ *for* $t \in \Sigma$.

The sketch of the proof is given in § 3. The complete proof of our result will be published elsewhere.

§ 3. Sketch of proof. We first restrict t to be real. We introduce sets $Q(s, L, k)$. Here k is any number satisfying $1-h<k<\min\{1-\alpha, \sigma\}$ and L is any positive number. A function $v(t)$, defined for $0\leqq t\leqq s$, is said to belong to $Q(s, L, k)$ if $v(0)=A_0^\alpha u_0$ and if $\|v(t_1)-v(t_2)\| \leqq L|t_1-t_2|^k$ for any t_1, t_2 in $[0, s]$. Then for sufficiently small positive s and for all $t \in [0, s]$, we get $\|v(t)\|<R$ for any function $v(t) \in Q(s, L, k)$. Hence the operator $A_v(t) = A(t, A_0^{-\alpha}v(t))$ is well defined for $t \in [0, s]$. Set $f_v(t) = f(t, A_0^{-\alpha}v(t))$ and $w_v(t) = A_0^\alpha w(t)$, where w is the unique solution of

$$\begin{cases} dw/dt + A_v(t)w = f_v(t), & t \in [0, s] \\ w(0) = u_0. \end{cases}$$

Then using the linear theory of Kato [1], we get $w_v \in Q(s, L, k)$ for sufficiently small s.

We set $F = Q(s, L, k)$ and define a transformation $w_v = Tv$ for $v \in F$. Then T maps F into itself. By some calculations we can prove the following key fact: If s is small enough, there exists $0<\theta<1$ such that for any $v_1, v_2 \in F$ the inequality $|||Tv_1 - Tv_2||| \leqq \theta |||v_1 - v_2|||$ holds (where $|||v||| = \sup_{0\leqq t\leqq s}\|v(t)\|$). So by the fixed point theorem there exists a unique point v in F such that $Tv=v$. Then $u=A_0^{-\alpha}v$ is a unique solution of (1.1), (1.2) which is continuously differentiable for $0<t \leqq s$, continuous for $0\leqq t\leqq s$.

Next we shall show that u is extensible analytically in t to a sector Σ. From (2.1) there are constants $C_4, \phi_1>0, T_1>0$ such that for $t \in \Sigma_1$, $w \in N$ and $|\theta|<\phi_1$ the resolvent of $e^{i\theta}A(t, A_0^{-\alpha}w)$ contains the left plane and

$$\|(\lambda - e^{i\theta}A(t, A_0^{-\alpha}w))^{-1}\| \leqq C_4(1+|\lambda|)^{-1} \qquad \mathrm{Re}\,\lambda\leqq 0,$$

where $\Sigma_1 \equiv \{t \in \boldsymbol{C}; |\arg t| < \phi_1, 0 \leqq |t| < T_1\}$. We let $\phi = \min\{\phi_0, \phi_1\}$, and in (1.1) and (1.2) we make the change of variable $t = \tau e^{i\theta}$, $\tau \in [0, T_1]$, $|\theta| < \phi$, so equations (1.1) and (1.2) become

$$\begin{cases} \partial v/\partial\tau + e^{i\theta}A(\tau e^{i\theta}, v)v = e^{i\theta}f(\tau e^{i\theta}, v), \\ v(0, e^{i\theta}) = u_0, \end{cases} \tag{3.1}$$

where $v(\tau, e^{i\theta}) = u(\tau e^{i\theta})$ and $u(t) = v(|t|, t/|t|)$.

We hold $|\theta| < \phi$ fixed and apply the argument about real t to equation (3.1). Then there exist T, $0 < T \leqq \min\{T_0, T_1\}$ and a unique solution $v(\tau, e^{i\theta})$ of (3.1) defined for $\tau \in [0, T]$, $|\theta| < \phi$. Let $\Sigma \equiv \{t \in \boldsymbol{C}; |\arg t| < \phi, 0 \leqq |t| < T\}$ and

$$\begin{cases} u(t) = v(|t|, t/|t|), \qquad t \in \Sigma \backslash \{0\}, \\ u(0) = u_0. \end{cases} \tag{3.2}$$

We can easily prove that u satisfies the conclusions of Theorem.

References

[1] T. Kato: Abstract evolution equations of parabolic type in Banach and Hilbert spaces. Nagoya Math. J., **5**, 93–125 (1961).

[2] F. J. Massey, III: Analyticity of solutoins of nonlinear evolution equations. J. Diff. Eqs., **22**, 416–427 (1976).