58. On τ Functions of a Class of Painlevé Type Equations. I*)

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1. The aim of the present note is to give the description of monodromy preserving deformation of a linear ordinary differential equation of the form

$$\mathcal{L}Y \equiv \left(x\frac{d}{dx}+L\frac{d}{dx}+Mx+N\right)Y=0 \tag{1}$$

in a Hamiltonian form and to establish transformation formulas of the associated 'τ functions' ([2]–[5]). Here the coefficients L, M and N are constant matrices of size r while Y can be a column vector as well as a square matrix of size r of functions of x. We assume that L (resp. M) has distinct eigenvalues which we write $-a_j$ (resp. $-c_j$), $j=1, \cdots, r$ so that $-L$ (resp. $-M$) is conjugate to the diagonal matrix $A=(a_j\delta_{jk})_{j,k=1,\cdots,r}$ (resp. $C=(c_j\delta_{jk})_{j,k=1,\cdots,r}$). Hereafter we shall normalize $-L=QAQ^{-1}$, $-M=C$ so that we can write

$$\mathcal{L}=Q(x-A)Q^{-1}\left(\frac{d}{dx}-C\right)-B=\left(\frac{d}{dx}-C\right)Q(x-A)Q^{-1}-B' \tag{2}$$

by setting $B=LM-N$, $B'=1+ML-N$. We have

$$B'=1+B-[QAQ^{-1}, C]. \tag{3}$$

We also set: $P=Q^{-1}B$, $E_j=(\delta_{kj}\delta_{k'j})_{k,k'=1,\cdots,r}$, and $B_j=QE_jP$. By writing our equation, $\mathcal{L}Y=0$, as

$$\frac{d}{dx}Y=(Q(x-A)^{-1}P+C)Y \tag{4}$$

and observing $(x-A)^{-1}=\sum_{j=1}^{r}(x-a_j)^{-1}E_j$, we see that (1) is equivalent to

$$\frac{d}{dx}Y=\left(\sum_{j=1}^{r}\frac{B_j}{x-a_j}+C\right)Y, \qquad \text{with } B_j \text{ of rank} \leq 1, \tag{5}$$

an equation with regular singularities at $x=a_1, \cdots, a_r$ and an irregular singularity of rank 1 at $x=\infty$. Note that the number of regular singularities is equal to the size r.

Conversely, suppose we are given an equation (5) with rank of $B_j \leq 1$ and $C=(c_j\delta_{jk})$ diagonal. Set $\lambda_j=\text{trace } B_j$ which is an eigenvalue of B_j, and define Q to be the matrix whose j-th column vector $[Q]_j$ is the eigenvector of B_j belonging to the eigenvalue λ_j: $B_j[Q]_j=\lambda_j[Q]_j$.

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Then we can set $B_j=QE_jP$, $j=1,\cdots,r$ and P consists of row eigenvectors of $B_1,\cdots,B_r$. Therefore the equation (5) is written as (4) which is equivalent to (1). Hence (1) and (5) are equivalent to each other.

We set $\Lambda=(\lambda_j\delta_{jk})$, $K=(\kappa_j\delta_{jk})$ where $\kappa_1,\cdots,\kappa_r$ denote diagonal elements of B $(=QP=\sum_{j=1}^r B_j)$ so that $\sum_{j=1}^r\kappa_j=\sum_{j=1}^r\lambda_j$ (for brevity we write $K=\operatorname{diag} B$ to mean that K is the diagonal part of B).

As will be discussed in the subsequent note II, the case $r=2$ corresponds to the deformation theory leading to the Painlevé equation of the fifth kind.

Our strategy is first to endow the coefficient matrices Q and P in (4) with a structure of canonical dynamical variables by defining their Poisson bracket by

(6) $$\{Q_{ij},P_{kl}\}=\delta_{il}\delta_{jk},\quad \{Q_{ij},Q_{kl}\}=0,\quad \{P_{ij},P_{kl}\}=0.$$

We denote by d the exterior differentiation with respect to A and C and define a 1 form ω by

(7) $$\omega(A,C)=\frac{1}{2}\sum_{i\neq j}(PQ)_{ij}(PQ)_{ji}d\log(a_i-a_j)+\sum_{i,j}Q_{ij}P_{ji}d(a_jc_i)+\frac{1}{2}\sum_{i\neq j}(QP)_{ij}(QP)_{ji}d\log(c_i-c_j).$$

Then the deformation equations, which describe dependence of the coefficient matrices $B_1,\cdots,B_r$ of the equation (5) or Q,P of (4) on the deformation parameters A and C, is given by

(8) $$dQ=\{Q,\omega\},\qquad dP=\{P,\omega\},$$
$$\text{i.e.}\quad dQ=Q\Theta^*+dC\cdot QA+CQdA+\Theta Q,$$
$$dP=-\Theta^*P-APdC-dA\cdot PC-P\Theta,$$

where Θ and Θ^* are defined by

(9) $$[\Theta,C]=[QP,dC],\quad \operatorname{diag}\Theta=0;\quad [A,\Theta^*]=[dA,PQ],\quad \operatorname{diag}\Theta^*=0.$$

Indeed, the linear equation (4) $\mathcal{L}Y=0$ with $\mathcal{L}$ of (2) and the equation

(10) $$dY=\Omega Y,\qquad \Omega=-QdA\cdot Q^{-1}\left(\frac{d}{dx}-C\right)+xdC+\Theta$$

are consistent under the conditions (8) because we have then

(11) $$d\mathcal{L}=\Omega^*\mathcal{L}-\mathcal{L}\Omega\quad\text{with}\quad \Omega^*=\Omega-[QdA\cdot Q^{-1},C]-[QAQ^{-1},dC].$$

Hence our 'Hamiltonian equations of motion' (8) describe the deformation of (5) under which the monodromy structure is preserved. If Q and P satisfy (8) the 1 form $\omega(A,C)$ of (7) is closed: $d\omega=0$. Hence the function $\tau(A,C)$ is well-defined uniquely up to a constant multiple by

(12) $$\omega(A,C)=d\log\tau(A,C),$$

which we call the 'τ function' of (8), in accordance with [4], [5].

The equation (5) admits a local solution $Y(x)$ at $x=A$ and a formal solution $Y^{(\infty)}(x)$ at $x=\infty$ of the following form:

(13) $$Y(x)=Q\sum_{n=0}^{\infty}Y_n\frac{(x-A)^{\Lambda+n}}{(\Lambda+n)!}e^{C(x-A)},\quad Y_0=1\quad (\text{at } x=A),$$

$$Y^{(\infty)}(x)=\sum_{n=0}^{\infty} Y_n^{(\infty)}(x-A)^{K-n}e^{Cx}, \quad Y_0^{(\infty)}=1 \quad (\text{at } x=\infty),$$

where the j-th column vector of $Y(x)$ is a local column vector solution at $x=a_j$ having the exponent λ_j. The coefficients Y_n and $Y_n^{(\infty)}$ are uniquely determined by

$$(14)_n \quad (Y_n-Q^{-1}[C, QY_{n-1}])(\Lambda+n)-[A, Y_{n+1}-Q^{-1}[C, QY_n]]=PQY_n,$$
$$Q^{-1}(Y_n^{(\infty)}(K-n)+[Y_{n+1}^{(\infty)}, C])-[A, Q^{-1}(Y_{n-1}^{(\infty)}(K-n+1)+[Y_n^{(\infty)}, C])]$$
$$=PY_n^{(\infty)}.$$

The solutions (13) thus determined are shown to automatically satisfy the deformation equation (10), and consistency of the suppositions $Y_0=1$ and $Y_0^{(\infty)}=1$ in (13) are verified also in the course.

We note that the diagonal parts of $(14)_0$ give

$$(15) \qquad \Lambda=\operatorname{diag} PQ, \qquad K=\operatorname{diag} QP,$$

while (8) together with (15) implies $d\Lambda=0$ and $dK=0$. Namely λ_j, κ_j $(j=1, \cdots, r)$ are the constants of integration of (8) as they should be.

It is manifest in (1)–(2) that the formal Laplace transformation

$$(16) \qquad \frac{d}{dx}\longmapsto y, \quad x\longmapsto -\frac{d}{dy}, \quad Y\longmapsto \hat{Y}$$

changes $\mathcal{L}Y=0$ into $((y-C)Q(-d/dy-A)-B'Q)Z=0$ with $Z=Q^{-1}\hat{Y}$; so we have

Theorem 1. *By the formal Laplace transformation* (16) *the equation* (4) *is transformed into*

$$(17) \quad \frac{d}{dy}Z=(\hat{Q}(y-C)^{-1}\hat{P}-A)Z \quad \textit{with} \quad \hat{Q}=Q^{-1}, \ \hat{P}=-B'Q, \ Z=Q^{-1}\hat{Y}.$$

In place of (15) *we have* (from (3))

$$(18) \qquad \operatorname{diag} \hat{Q}\hat{P}=-(1+\Lambda), \qquad \operatorname{diag} \hat{P}\hat{Q}=-(1+K).$$

Namely the transformation means the replacement:

$$(19) \qquad (Q, P, A, C)\longmapsto(\hat{Q}, \hat{P}, C, -A).$$

We claim

Theorem 2. *$\hat{Q}$ and $\hat{P}$ constitute canonical transforms of Q and P, and the deformation equations* (8) *stay invariant under the transformation* (19).

Since the same statement as Theorem 2 is obviously true with the transformation:

$$(20) \qquad (Q, P, A, C)\longmapsto(P, -Q, C, -A),$$

and since (19) is the composition of (20) and

$$(21) \qquad (Q, P, A, C)\longmapsto(-\hat{P}, \hat{Q}, A, C),$$

we see that Theorem 2 is reduced to the corresponding statement with (21), for which we give a proof below.

2. Let us write (21) as $(Q, P)\mapsto(Q', P')$ by setting $Q'=-\hat{P}$ and $P'=\hat{Q}$, or more explicitly

$$(22) \qquad Q'=(1+QP-[QAQ^{-1}, C])Q, \qquad P'=Q^{-1}$$

whose inverse transformation is given by

(23) $$Q=P'^{-1}, \qquad P=P'(-1+Q'P'+[P'^{-1}AP', C]).$$

For any expression $F=f(Q, P, A, C)$ we shall write $F'=f(Q', P', A, C)$; for example for $B=QP$ we write $B'=Q'P'$, in coincidence with (3).

From (7) and (22) we have the identity

(24) $$\omega'-\omega=\operatorname{trace}(QAQ^{-1}dC+Q^{-1}CQdA),$$

and using this and (22) we also have

(25) $$(\operatorname{trace} P'dQ'-\omega')-(\operatorname{trace} PdQ-\omega)=dW, \qquad \text{with}$$
$$W=W(Q, Q', A, C)=\operatorname{trace} Q^{-1}(Q'-CQA)+\log\det Q.$$

Because of the independence of $Q=P'^{-1}$ and Q' (25) shows that the transformation (22) is a canonical transformation. Therefore if Q and P satisfy (8) then Q' and P' satisfy the same equations.

Now (18) reads

(26) $$\Lambda'=1+\Lambda, \qquad K'=1+K;$$

namely, the constants of integration λ_j's and κ_j's undergo simultaneous increase by 1 under this transformation (22) of solutions of (8).

We now introduce the following transformation (an example of Schlesinger's transformation [1]):

(27) $$Y'=Q(x-A)Q^{-1}Y,$$

whose inverse is given by

(28) $$Y=P'^{-1}Q'^{-1}\left(\frac{d}{dx}-C\right)Y'$$

by virtue of the second expression of $\mathcal{L}$ in (2). We have now

Theorem 3. *By the Schlesinger transformation* (27) *the equations* (1) *and* (10) *are transformed into* $\mathcal{L}'Y'=0$ *and* $dY'=\Omega'Y'$. *More specifically, the equation* (4) *and the solutions* (13) *are transformed respectively into*

(29) $$\frac{d}{dx}Y'=(Q'(x-A)^{-1}P'+C)Y'=\left(\sum_{j=1}^{r}\frac{B'_j}{x-a_j}+C\right)Y',$$

and

(30) $$Y'(x)=Q'\sum_{n=0}^{\infty}Y'_n\frac{(x-A)^{1+\Lambda+n}}{(1+\Lambda+n)!}e^{C(x-A)} \qquad (\text{at } x=A),$$
$$Y^{(\infty)\prime}(x)=\sum_{n=0}^{\infty}Y_n^{(\infty)\prime}(x-A)^{1+K-n}e^{Cx} \qquad (\textit{at } x=\infty),$$

where Q', P' *are given by* (22), *and* $Y'_n, Y_n^{(\infty)\prime}$ *by*

(31) $$Y'_n=Q'^{-1}Q(Y_n(1+\Lambda+n)-[A, Y_{n+1}]), \qquad Y'_0=1;$$
$$Y_n^{(\infty)\prime}=Y_n^{(\infty)}-Q[A, Q^{-1}Y_{n-1}^{(\infty)}], \qquad Y_0^{(\infty)\prime}=1.$$

We now proceed to the transformation formula to the τ function. We get $d\log\det Q=\operatorname{trace}(QAQ^{-1}dC+Q^{-1}CQdA)$ by (8). Comparing this with (24) we see $d\log\tau'-d\log\tau=d\log\det Q$, and obtain the following formula.

Theorem 4. *We have, by suitably normalizing constant factors of* τ *functions,*

(32) $$\frac{\tau'}{\tau}=\det Q.$$

We denote by $Q^{(n)}, P^{(n)}, F^{(n)}=f(Q^{(n)}, P^{(n)}, A, C)$ (resp. $Y^{(n)}$) the transforms of $Q, P, F=f(Q, P, A, C)$ (resp. Y) by (22) (resp. (27)) iterated n times; whence $\operatorname{diag} P^{(n)}Q^{(n)}=n+\Lambda$, $\operatorname{diag} Q^{(n)}P^{(n)}=n+K$.

We define an $nr\times nr$ matrix R_n as follows.

(33) $$R_n=(R_{i,j})_{i,j=0,1,\cdots,n-1}$$

where $R_{i,j}$ are $r\times r$ matrices given by

(34) $$R_{00}=Q,\quad R_{0,j+1}=CR_{0j},\quad R_{i+1,0}=R_{i0}A,$$
$$R_{i+1,j}=R_{ij}A+jR_{i,j-1}+\sum_{k=1}^{j-1}R_{i,j-1-k}PC^kQ \qquad (j=1,2,3,\cdots).$$

For example

$$R_3=\begin{bmatrix} Q & CQ & C^2Q \\ QA & CQA+Q+QPQ & C^2QA+2CQ+CQPQ+QPCQ \\ QA^2 & CQA^2+2QA+QPQA+QAPQ & R_{2,2}\end{bmatrix},$$

$$R_{2,2}=C^2QA^2+4CQA+2Q+CQPQA+CQAPQ+QPCQA+QAPCQ$$
$$+3QPQ+QPQPQ.$$

Theorem 5. *We have, by using the same normalization as in* (32),

(35) $$\frac{\tau^{(n)}}{\tau^{(0)}}=\det R_n.$$

We can derive from the definition (34)

(36) $$R_n$$
$$=\begin{bmatrix} 1 & & & \\ QAQ^{-1} & \ddots & & \\ \vdots & \ddots & 1 & \\ QA^{n-1}Q^{-1} & \cdots & QAQ^{-1} & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & Q^{(n-2)}A(Q^{(n-2)})^{-1} & 1 \end{bmatrix}\begin{bmatrix} Q & & & \\ & Q' & & \\ & & \ddots & \\ & & & Q^{(n-1)} \end{bmatrix}$$
$$\times\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & (Q^{(n-2)})^{-1}CQ^{(n-2)} \\ & & & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & Q^{-1}CQ & \cdots & Q^{-1}C^{n-1}Q \\ & 1 & \ddots & \vdots \\ & & \ddots & Q^{-1}CQ \\ & & & 1 \end{bmatrix},$$

whence we have $\det R_n=\det Q\,\det Q'\cdots\det Q^{(n-1)}$ which together with Theorem 4 implies Theorem 5.

The inverse transformation (28) is rewritten as $Q^{-1}Y=Q'^{-1}(d/dx-C)Q'\cdot Q'^{-1}Y'$ which through the Laplace transformation (16) reads

(37) $$Z=\hat{Q}'(y-C)(\hat{Q}')^{-1}Z' \quad\text{or}\quad Z^{(n-1)}=\hat{Q}^{(n)}(y-C)(\hat{Q}^{(n)})^{-1}Z^{(n)}$$

where we write $\hat{Q}^{(n)}=(Q^{(n)})^{-1}=P^{(n+1)}$, $Z^{(n)}=(Q^{(n)})^{-1}\widehat{Y^{(n)}}$ in accordance with the convention. It is now easy to see that the canonical transformation (19) induces the transformation of associated quantities

(38) $$(Q^{(n)}, P^{(n)}, \omega^{(n)}, \cdots)\longmapsto((-)^nP^{(1-n)}, (-)^{n-1}Q^{(1-n)}, \omega^{(1-n)}, \cdots)$$

while (20) induces

(39) $(Q^{(n)}, P^{(n)}, \omega^{(n)}, \cdots) \longmapsto ((-)^n P^{(-n)}, (-)^{n-1} Q^{(-n)}, \omega^{(-n)}, \cdots)$.

Specifically, if we set $Q^{(n)} = q_n(Q, P, A, C)$ and $P^{(n)} = p_n(Q, P, A, C)$ then we have $(-)^n P^{(-n)} = q_n(P, -Q, C, -A)$ and $(-)^{n-1} Q^{(-n)} = p_n(P, -Q, C, -A)$. Similarly we get for $n \geq 0$

(40) $$(-)^{r(n(n-1)/2)} \frac{\tau^{(-n)}}{\tau^{(0)}} = \det R_n^*, \quad \text{with} \quad R_n^* = R_n|_{(Q,P,A,C) \mapsto (P,-Q,C,-A)}.$$

Let $I = (i_1, \cdots, i_k)$, $I' = (i_{k+1}, \cdots, i_r)$ be ordered subsets of $\{1, 2, \cdots, r\}$ complementary to each other. We denote by $M_{I,J} = M_{(i_1,\cdots,i_k),(j_1,\cdots,j_k)}$ the minor of size k of a matrix M. (36) tells that $P^{(n)} = (Q^{(n-1)})^{-1}$ is in the last $r \times r$ block of R_n^{-1}. Using this fact and the formula: $(M^{-1})_{I,J} = ((-)^{|I|+|J|}/\det M) M_{J',I'}$, $|I| = i_1 + \cdots + i_k$, we have

(41) $$\tau^{(n)} \cdot P_{I,J}^{(n)} = (-)^{|I|+|J|} \tau^{(0)} \cdot (R_n)_{(1,\cdots,(n-1)r,(n-1)r+J'),(1,\cdots,(n-1)r,(n-1)r+I')},$$
$$\tau^{(n)} \cdot Q_{I,J}^{(n)} = \tau^{(0)} \cdot (R_{n+1})_{(1,\cdots,nr,nr+I),(1,\cdots,nr,nr+J)},$$

where $l + I = (l + i_1, \cdots, l + i_k)$. Likewise we have

(42) $$\tau^{(-n)} \cdot Q_{I,J}^{(-n)} = (-)^{|I|+|J|} \tau^{(0)} \cdot (R_n^*)_{(1,\cdots,(n-1)r,(n-1)r+J'),(1,\cdots,(n-1)r,(n-1)r+I')},$$
$$\tau^{(-n)} \cdot P_{I,J}^{(-n)} = \tau^{(0)} \cdot (R_{n+1}^*)_{(1,\cdots,nr,nr+I),(1,\cdots,nr,nr+J)}.$$

From these identities we conclude

Theorem 6. *$\tau^{(n)} P_{I,J}^{(n)}$ and $\tau^{(n)} Q_{I,J}^{(n)}$ ($n = 0, \pm 1, \pm 2, \cdots; k = 0, 1, \cdots, r$ with $k = \sharp(I) = \sharp(J)$) are all (multi-valued) holomorphic outside $\bigcap_{j=-\infty}^{+\infty} S_j$, where S_j is the union of the singularities of $Q^{(j)}, P^{(j)}$ and $\tau^{(j)}$.*

Note that both $\tau^{(n)} P_{I,J}^{(n)}$ and $\tau^{(n)} Q_{I,J}^{(n)}$ reduce to $\tau^{(n)}$ when $k = 0$.

From (3): $Q'P' = 1 + QP - [QAQ^{-1}, C]$ and its variant: $P'Q' = 1 + PQ - [A, Q^{-1}CQ]$ we obtain

Corollary 7. *If τ and τ' have no common divisor outside $\bigcap_{j=-\infty}^{+\infty} S_j$, then $\tau^{(n)} Q^{(n)} P^{(n)}$ and $\tau^{(n)} P^{(n)} Q^{(n)}$ are also holomorphic outside $\bigcap_{j=-\infty}^{+\infty} S_j$.*

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