57. On the Strong Convergence of the Cèsaro Means of Contractions in Banach Spaces

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1. Introduction. Throughout this paper X denotes a uniformly convex Banach space and C is a nonempty closed convex subset of X. A mapping $T: C \to C$ is called a contraction on C, or $T \in \mathrm{Cont}(C)$ if $\|Tx-Ty\| \leqq \|x-y\|$ for every $x, y \in C$. A family $\{T(t); t \geqq 0\}$ of mappings from C into itself is called a contraction semi-group on C if $T(0) = I$ (the identity on C), $T(t+s)=T(t)T(s)$, $T(t) \in \mathrm{Cont}(C)$ for $t, s \geqq 0$ and $\lim_{t\to 0+} T(t)x = x$ for every $x \in C$. The set of fixed points of a mapping T will be denoted by $F(T)$.

The purpose of this paper is to prove the following (nonlinear) mean ergodic theorems.

Theorem 1. *Let $T \in \mathrm{Cont}(C)$, $x \in C$ and $F(T) \neq \emptyset$. If $\lim_{n\to\infty} \|T^n x - T^{n+i}x\|$ exists uniformly in $i=1, 2, \cdots$, then there exists an element $y \in F(T)$ such that*

$$\text{(the strong limit)} \lim_{n\to\infty} n^{-1} \sum_{i=0}^{n-1} T^{i+k}x = y \text{ uniformly in } k=0, 1, 2, \cdots.$$

Theorem 2. *Let $\{T(t); t \geqq 0\}$ be a contraction semi-group on C, $x \in C$ and $\bigcap_{t>0} F(T(t)) \neq \emptyset$. If $\lim_{t\to\infty} \|T(t)x - T(t+h)x\|$ exists uniformly in $h>0$, then there exists an element $y \in \bigcap_{t>0} F(T(t))$ such that*

$$\lim_{t\to\infty} t^{-1} \int_0^t T(s+h)x\, ds = y \qquad \text{uniformly in } h \geqq 0.$$

These results have been known in Hilbert space (cf. [1, 2]).

2. Proofs of Theorems. For a given $T \in \mathrm{Cont}(C)$ we set $S_n = n^{-1}(I+T+\cdots+T^{n-1})$ for every $n \geqq 1$. We start with the following

Lemma 1. *Let $T \in \mathrm{Cont}(C)$, $x \in C$ and $F(T) \neq \emptyset$. Suppose that*

(*) $\lim_{n\to\infty} \|T^n x - T^{n+i}x\|$ *exists uniformly in $i=1, 2, \cdots$.*

Then we have

(1) $\lim_{n,m\to\infty} \|2^{-1}(S_n T^{l+n}x + S_m T^{l+m}x) - T^l(2^{-1}S_n T^n x + 2^{-1} S_m T^m x)\| = 0$ *uniformly in $l=1, 2, \cdots$. In particular,*

(2) $\lim_{n\to\infty} \|S_n T^{l+n}x - T^l S_n T^n x\| = 0$ *uniformly in $l=1, 2, \cdots$.*

Proof. Take an $f \in F(T)$ and an $r>0$ with $r \geqq \|x-f\|$, and set $D = \{z \in X; \|z-f\| \leqq r\} \cap C$ and $U = T|_D$ (the restriction of T to D). Since D is bounded closed convex and $U \in \mathrm{Cont}(D)$, by virtue of [4, Theorem

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2.1] (cf. [3, Lemma 1.1]) there exists a strictly increasing, continuous convex function $\gamma:[0,\infty)\to[0,\infty)$ with $\gamma(0)=0$ such that

$$\left\|U^l\Big(\sum_{i=1}^k \lambda_i x_i\Big)-\sum_{i=1}^k \lambda_i U^l x_i\right\| \leqq \gamma^{-1}\Big(\max_{1\leqq i,j\leqq k}[\|x_i-x_j\|-\|U^l x_i-U^l x_j\|]\Big)$$

for any $\lambda_1,\cdots,\lambda_k\geqq 0$ with $\lambda_1+\cdots+\lambda_k=1$, any $x_1,\cdots,x_k\in D$ and any $k,l\geqq 1$. Consequently

$$\left\|T^l\Big(\sum_{i=0}^{n-1}\lambda_i x_i+\sum_{i=1}^{m-1}\mu_i y_i\Big)-\Big(\sum_{i=0}^{n-1}\lambda_i T^l x_i+\sum_{i=0}^{m-1}\mu_i T^l y_i\Big)\right\| \leqq \gamma^{-1}(\max\{\|x_i-x_j\|-\|T^l x_i-T^l x_j\|,\ \|x_i-y_p\|-\|T^l x_i-T^l y_p\|,\ \|y_p-y_q\|-\|T^l y_p-T^l y_q\|;\ 0\leqq i,j\leqq n-1,\ 0\leqq p,q\leqq m-1\})$$

for any $\lambda_i,\mu_i\geqq 0$ with $\sum_{i=0}^{n-1}\lambda_i+\sum_{i=0}^{m-1}\mu_i=1$, any $x_i,y_i\in D$ and any $n,m\geqq 1$, $l\geqq 0$. Using this inequality with $\lambda_i=1/2n$, $x_i=T^{i+n}x$ for $0\leqq i\leqq n-1$ and $\mu_i=1/2m$, $y_i=T^{i+m}x$ for $0\leqq i\leqq m-1$, we obtain

$$(3)\quad \|T^l(2^{-1}S_nT^nx+2^{-1}S_mT^mx)-(2^{-1}S_nT^{l+n}x+2^{-1}S_mT^{l+m}x)\| \leqq \gamma^{-1}(\max\{\|T^{i+n}x-T^{j+n}x\|-\|T^{l+i+n}x-T^{l+j+n}x\|,\ \|T^{i+n}x-T^{p+m}x\| -\|T^{l+i+n}x-T^{l+p+m}x\|,\ \|T^{p+m}x-T^{q+m}x\| -\|T^{l+p+m}x-T^{l+q+m}x\|;\ 0\leqq i,j\leqq n-1,\ 0\leqq p,q\leqq m-1\})$$

for any $n,m\geqq 1$ and $l\geqq 0$.

For any $\varepsilon>0$ choose a $\delta>0$ such that $\gamma^{-1}(\delta)<\varepsilon$. By (*) there exists a positive integer N such that $\beta(i)\leqq\|T^nx-T^{n+i}x\|<\beta(i)+\delta$ for every $i\geqq 0$ and $n\geqq N$, where $\beta(i)=\lim_{n\to\infty}\|T^nx-T^{n+i}x\|$. Hence if $n,m\geqq N$ then

$$\|T^{i+n}x-T^{j+m}x\|-\|T^{l+i+n}x-T^{l+j+m}x\| <\beta(|j+m-i-n|)+\delta-\beta(|j+m-i-n|)=\delta \qquad \text{for every } i,j\geqq 0.$$

Combining this with (3), we obtain that if $n,m\geqq N$ then

$$\|T^l(2^{-1}S_nT^nx+2^{-1}S_mT^mx)-(2^{-1}S_nT^{l+n}x+2^{-1}S_mT^{l+m}x)\|\leqq\gamma^{-1}(\delta)<\varepsilon$$

for any $l\geqq 0$. Thus (1) holds true. Taking $m=n$ in (1), we have (2).

Q.E.D.

We note that for every sequence $\{x_n\}$ in X the following equality holds good: For any $l,p\geqq 1$ and $k\geqq 0$

$$(4)\quad l^{-1}\sum_{i=0}^{l-1}x_{i+k}=l^{-1}\sum_{i=0}^{l-1}\Big(p^{-1}\sum_{j=0}^{p-1}x_{i+j+k}\Big)+(lp)^{-1}\sum_{i=1}^{p-1}(p-i)(x_{i+k-1}-x_{i+k+l-1}).$$

Lemma 2. *Let $T\in \mathrm{Cont}(C)$, $x\in C$ and $F(T)\neq\emptyset$. If (*) is satisfied, then $\{\|S_nT^nx-f\|\}$ is convergent for every $f\in F(T)$.*

Proof. Let $f\in F(T)$ and $\alpha_n=\sup_{j\geqq 0}\|S_nT^{n+j}x-T^jS_nT^nx\|$ for $n\geqq 1$. Since

$$S_{n+m}T^{n+m}x=(n+m)^{-1}\sum_{i=0}^{n+m-1}(S_nT^{n+m+i}x-T^{m+i}S_nT^nx) +(n+m)^{-1}\sum_{i=0}^{n+m-1}T^{m+i}S_nT^nx +[n(n+m)]^{-1}\sum_{i=1}^{n-1}(n-i)[T^{n+m+i-1}x-T^{2(n+m)+i-1}x]$$

by (4),

$$\|S_{n+m}T^{n+m}x-f\|\leqq\alpha_n+(n+m)^{-1}\sum_{i=0}^{n+m-1}\|T^{m+i}S_nT^nx-f\|$$
$$+[n(n+m)]^{-1}\sum_{i=1}^{n-1}(n-i)\|T^{n+m+i-1}x-T^{2(n+m)+i-1}x\|$$
$$\leqq\alpha_n+\|S_nT^nx-f\|+(n-1)\|x-f\|/(n+m)$$

for $n, m\geqq 1$. Letting $m\to\infty$, we have

$$\limsup_{m\to\infty}\|S_mT^mx-f\|\leqq\alpha_n+\|S_nT^nx-f\| \qquad \text{for } n\geqq 1.$$

Therefore $\limsup_{m\to\infty}\|S_mT^mx-f\|\leqq\liminf_{n\to\infty}\|S_nT^nx-f\|$ because $\lim_{n\to\infty}\alpha_n=0$ by (2). Q.E.D.

Lemma 3. *Let $T\in\mathrm{Cont}(C)$, $x\in C$ and $F(T)\neq\emptyset$. If (*) is satisfied, then there exists an element $y\in F(T)$ such that $\lim_{n\to\infty}S_nT^{n+k}x=y$ uniformly in $k\geqq 0$.*

Proof. Take an $f\in F(T)$ and set $u_n=S_nT^nx-f$ for $n\geqq 1$. By Lemma 2, $d=\lim_{n\to\infty}\|u_n\|$ exists. Since $\|u_{n+1}-u_n\|\to 0$ as $n\to\infty$, we have

(5) $\qquad \lim_{n\to\infty}\|u_n+u_{n+i}\|=2d \qquad$ for every $i\geqq 0$.

We now show that $\{S_nT^nx\}$ is strongly convergent to an element of $F(T)$. Since $S_{n+k}T^{n+k}x=(n+k)^{-1}\sum_{i=0}^{n+k-1}S_nT^{n+k+i}x+v(n,k)$ by (4) and $\|v(n,k)\|\leqq(n-1)\|x-f\|/(n+k)$, where

$$v(n,k)=[n(n+k)]^{-1}\sum_{i=1}^{n-1}(n-i)[T^{n+k+i-1}x-T^{2(n+k)+i-1}x],$$

we have

$$\|u_{n+k}+u_{m+k}\|=\Big\|(n+k)^{-1}\sum_{i=0}^{n+k-1}(S_nT^{n+k+i}x+S_mT^{m+k+i}x-2f)$$
$$+[(n-m)/(m+k)(n+k)]\sum_{i=0}^{n+k-1}(S_mT^{m+k+i}x-f)$$
$$+(m+k)^{-1}\sum_{i=n+k}^{m+k-1}(S_mT^{m+k+i}x-f)+v(n,k)+v(m,k)\Big\|$$
$$\leqq(2/(n+k))\sum_{i=0}^{n+k-1}\|2^{-1}(S_nT^{n+k+i}x+S_mT^{m+k+i}x)-f\|$$
$$+2(m-n)\|x-f\|/(m+k)$$
$$+[(n-1)/(n+k)+(m-1)/(m+k)]\|x-f\|$$

for $m\geqq n\geqq 1$ and $k\geqq 0$. Combining this with $\|2^{-1}(S_nT^{n+k+i}x+S_mT^{m+k+i}x)-f\|\leqq\alpha_{n,m}+\|2^{-1}(S_nT^nx+S_mT^mx)-f\|$, where $\alpha_{n,m}=\sup_{l\geqq 0}\|2^{-1}(S_nT^{n+l}x+S_mT^{m+l}x)-T^l(2^{-1}S_nT^nx+2^{-1}S_mT^mx)\|$, we obtain

$$\|u_{n+k}+u_{m+k}\|\leqq 2\alpha_{n,m}+\|u_n+u_m\|+[(n-1)/(n+k)$$
$$+(m-1)/(m+k)]\|x-f\|+2(m-n)\|x-f\|/(m+k)$$
$$\text{for } m\geqq n\geqq 1 \text{ and } k\geqq 0.$$

Letting $k\to\infty$, it follows from (5) that

$$2d\leqq 2\alpha_{n,m}+\|u_n+u_m\|\leqq 2\alpha_{n,m}+\|u_n\|+\|u_m\|$$

for every $n, m\geqq 1$. Since $\lim_{n,m\to\infty}\alpha_{n,m}=0$ by Lemma 1, we have that $\lim_{n,m\to\infty}\|u_n+u_m\|=2d$. By uniform convexity of X and $\lim_{n\to\infty}\|u_n\|=d$, $\lim_{n,m\to\infty}\|S_nT^nx-S_mT^mx\|=\lim_{n,m\to\infty}\|u_n-u_m\|=0$, whence $\{S_nT^nx\}$ converges strongly. Put $y=\lim_{n\to\infty}S_nT^nx$. By (2), $\|S_nT^nx-TS_nT^nx\|$

$\leqq 2\|x-f\|/n+\|S_nT^{n+1}x-TS_nT^nx\|\to 0$ as $n\to\infty$ and hence $y\in F(T)$.

Finally, by (2) again,

$$\sup_{k\geqq 0}\|S_nT^{n+k}x-y\|\leqq\sup_{k\geqq 0}[\|S_nT^{n+k}x-T^kS_nT^nx\| + \|T^kS_nT^nx-y\|]\leqq\sup_{k\geqq 0}\|S_nT^{n+k}x-T^kS_nT^nx\|+\|S_nT^nx-y\|\to 0$$

as $n\to\infty$. Q.E.D.

Proof of Theorem 1. By virtue of Lemma 3, there exists an element $y\in F(T)$ such that $\lim_{n\to\infty}S_nT^{n+k}x=y$ uniformly in $k\geqq 0$. Therefore, for any $\varepsilon>0$ there exists a positive integer N such that $\|S_NT^{N+j}x-y\|<\varepsilon$ for all $j\geqq 0$. Since

$$S_nT^kx=n^{-1}\sum_{i=0}^{n-1}S_NT^{k+i}x+(nN)^{-1}\sum_{i=1}^{N-1}(N-i)(T^{k+i-1}x-T^{k+i+n-1}x)$$

by (4), if $n>N$ then

$$\begin{aligned}\|S_nT^kx-y\|&\leqq n^{-1}\sum_{i=0}^{n-1}\|S_NT^{k+i}x-y\|+(N-1)\|x-y\|/n\\ &\leqq n^{-1}\sum_{i=0}^{N-1}\|S_NT^{k+i}x-y\|+n^{-1}\sum_{i=N}^{n-1}\|S_NT^{k+i}x-y\|\\ &\quad+(N-1)\|x-y\|/n\\ &\leqq N\|x-y\|/n+\varepsilon+(N-1)\|x-y\|/n \qquad \text{for all } k\geqq 0.\end{aligned}$$

Hence $\sup_{k\geqq 0}\|S_nT^kx-y\|\to 0$ as $n\to\infty$. Q.E.D.

Remarks. 1) Let $T\in \mathrm{Cont}(C)$ and $x\in C$. If $\{T^nx\}$ has a convergent subsequence, then condition (*) is satisfied (cf. [2, Theorem 2.4]).

2) Let X be a Hilbert space, and let $T\in\mathrm{Cont}(C)$. If T is odd, then condition (*) is satisfied for every $x\in C$ (cf. [1, 2]).

Proof of Theorem 2. Similarly as in the proof of the preceding lemmas, we have the following (a)–(c):

(a) $\lim_{s,t\to\infty}\|2^{-1}(S_tT(t+h)x+S_sT(s+h)x)-T(h)(2^{-1}S_tT(t)x+2^{-1}S_sT(s)x)\|=0$ uniformly in $h>0$;

(b) $\lim_{t\to\infty}\|S_tT(t)x-f\|$ exists for every $f\in F\equiv\bigcap_{t>0}F(T(t))$;

(c) there exists an element $y\in F$ such that $\lim_{t\to\infty}S_tT(t+h)x=y$ uniformly in $h\geqq 0$; where $S_tz=t^{-1}\int_0^t T(s)z\,ds$ for $z\in C$ and $t>0$.

To prove (b) and (c) we use the following equality instead of (4);

$$t^{-1}\int_0^t T(\xi+h)x\,d\xi=t^{-1}\int_0^t[s^{-1}\int_0^s T(\xi+\eta+h)x\,d\eta]d\xi + (ts)^{-1}\int_0^s(s-\eta)[T(\eta+h)x-T(\eta+t+h)x]d\eta$$

for t, $s>0$ and $h\geqq 0$. Now, the same argument in the proof of Theorem 1 implies that $\lim_{t\to\infty}S_tT(h)x=y$ uniformly in $h\geqq 0$. Q.E.D.

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