6. On a Certain Integral Equation of Fredholm of the First Kind and a Related Singular Integral Equation

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1. It is the purpose of this paper to give an explicit formulation for the solution of an integral equation of Fredholm of the first kind

$$(1)\qquad \int_l \left\{\frac{-1}{2\pi} ln\,|x-y| + C_0 + C_1(x-y)^2 + C_2(x-y)^2 ln\,|x-y|\right\}\tau(y)dy = g(x)$$

where $l=\bigcup_{j=1}^{\nu} l_j$ is the union of a finite number of bounded intervals $l_j=(a_j, b_j)$, $(a_j<b_j<a_{j+1};\, j=1,2,\cdots,\nu,\, a_{\nu+1}=\infty)$, C's are known complex valued constants, and $g(x)$ is a given, continuously differentiable function. The unknown function $\tau(x)$ is assumed to have a singularity of $O(1/\sqrt{x-c})$ at each of the end points $c=a_j$ and $c=b_j$ and otherwise is continuous.

In his previous paper [1], one of the authors showed that the Dirichlet problem for the Helmholtz equation for an open boundary l is equivalent to that of solving the integral equation

$$(2)\qquad \int_l \psi(x, y)\tau(y)dy = g(x)$$

where $\psi(x, y)=(1/4i)H_0^{(2)}\,(k\,|x-y|)$ and $H_0^{(2)}$ is the second kind Hankel function of the zero-th order. If the "length" of l, or $(b_\nu - a_1)$, is such that $k^4(b_\nu-a_1)^4=O(1)$ holds for a given "wave number" k, the kernel $\psi(x, y)$ of (2) is well approximated by that of (1), and (1) is an approximation of (2). If a solution of (1) is obtained, a higher order approximation to the solution of (2) is available by successive approximations.

On the other hand, after differentiation with respect to x, (1) is converted to the singular integral equation

$$(3)\qquad \frac{1}{\pi i}\int_l \left\{\frac{1}{y-x} - A(y-x) - B(y-x)ln\,|x-y|\right\}\tau(y)dy = h(x)$$

where $A=2\pi(2C_1+C_2)$, $B=4\pi C_2$ and $h(x)=(2/i)(dg(x)/dx)$, and the integral is taken in the sense of Cauchy's principal value [1].

There are many works on singular integral equations [2], [3], however, to the best knowledge of the author, an equation like (3), whose kernel has a Cauchy type singularity and a log singularity simultaneously, has never been solved explicitly.

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In the following sections, an explicit solution of (3) will be derived by converting it to a Hilbert problem in a complex plane, from which will derive the solution of (1) as well.

2. Let $\Phi(z)$ be a function of a complex variable z defined by

$$\Phi(z)=\frac{1}{2\pi i}\sum_{j=1}^{\nu}\int_{a_j}^{b_j}\left\{\frac{1}{y-z}-A(y-z)-B(y-z)ln\left(\frac{y-z}{a_j-z}\right)\right\}\tau(y)dy$$

and let $\Phi^{\pm}(x)$ be the limiting values of $\Phi(z)$ when $z=x+iy$ tends to a point x on the x-axis in such a manner that $y\to 0\pm$. Then, it is easy to see that

(i) $\Phi(z)$ is analytic everywhere except for $z=\infty$ and $z\in\bar{l}$, where $\bar{l}$ is the closure of l.

(ii) $\Phi(z)=O(|z|)$ when $|z|\to\infty$.

(iii) $\Phi(z)=O(1/\sqrt{(z-c)})$ when $z\to c=a_j$ or b_j.

(iv) $\Phi^+(x)+\Phi^-(x)=h(x)+\sum_{j=1}^{\nu}(P_{j0}+P_{j1}x)\,ln|a_j-x|$, $(x\in l)$

(v) $\Phi^+(x)-\Phi^-(x)=\tau(x)-B\int_x^{b_j}(y-x)\tau(y)dy$, $(x\in l_j)$

where P_{j0} and P_{j1} are constants, if one notes that

$$(4)\qquad \left(\frac{1}{2\pi i}\int_l\frac{\tau(y)}{y-z}dy\right)^{\pm}=\pm\frac{1}{2}\tau(x)+\frac{1}{2\pi i}\int_l\frac{\tau(y)}{y-x}dy,\qquad (x\in l)$$

and

$$\left(\frac{1}{2\pi i}\int_{l_i}(y-z)ln\left(\frac{y-z}{a_i-z}\right)\tau(y)dy\right)^{\pm}$$

$$=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{l_i}(y-x)ln\left|\frac{y-x}{a_i-x}\right|\tau(y)dy, & (x\in l_j,\ i\neq j)\\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{l_j}(y-x)ln\left|\frac{y-x}{a_j-x}\right|\tau(y)dy\pm\frac{1}{2}\int_x^{b_j}(y-x)\tau(y)dy, & (x\in l_j,\ i=j).\end{cases}$$

On the other hand, if $\Psi(z)$ is defined by

$$\Psi(z)=\frac{X(z)}{2\pi i}\int_l\frac{1}{y-z}\frac{dy}{X(y)}\left\{h(y)+\sum_{j=1}^{\nu}(P_{j0}+P_{j1}y)ln|a_j-y|\right\}+\frac{X(z)}{2}\sum_{m=0}^{\nu+1}q_m z^m$$

where the q's are constants to be determined, and $X(z)$ is a function defined by

$$X(z)=1\Big/\sqrt{\prod_{j=1}^{\nu}(z-a_j)(z-b_j)}$$

whose limiting values at x on l are denoted as

$$X(x)=X^+(x)=-X^-(x),$$

then, it is not difficult to see that

(vi) $\Psi(z)$ is analytic everywhere except for $z=\infty$ and $z\in\bar{l}$.

(vii) $\Psi(z)=O(|z|)$ when $|z|\to\infty$.

(viii) $\Psi(z)=O(1/\sqrt{z-c})$ when $z\to c=a_j$ or b_j.

(ix) $\Psi^+(x)+\Psi^-(x)=h(x)+\sum_{j=1}^{\nu}(P_{j0}+P_{j1}x)ln|a_j-x|$.

Note that (ix), and (x) below as well, are obtained with the help of formula (4) applied to $\Psi(z)$.

As a consequence, $\Psi(z)$ is shown to be identical with $\Phi(z)$ if constants q's are suitably chosen (and this will be done later), because $\{\Phi(z)-\Psi(z)\}/X(z)$ is proved, by virtue of the properties of $\Phi(z)$ and $\Psi(z)$ mentioned above, to be a polynomial of order $\nu+1$. Therefore, we have $\Phi^+(x)-\Phi^-(x)=\Psi^+(x)-\Psi^-(x)$. While, from (v), the left hand member of this equation is

$$\tau(x)-B\int_x^{b_j}(y-x)\tau(y)dy,$$

and the right hand member is shown to be

(x) $\Psi^+(x)-\Psi^-(x)=f(x)$

where we have set

$$(5)\qquad f(x)=\frac{X(x)}{\pi i}\int_l \frac{1}{y-x}\frac{dy}{X(y)}\left\{h(y)+\sum_{j=1}^{\nu}(P_{j0}+P_{j1}y)ln\,|a_j-y|\right\}+X(x)\sum_{m=0}^{\nu+1}q_m x^m.$$

That is, we have shown that a solution $\tau(x)$ must satisfy the following integral equations of Volterra of the second kind on l_j,

$$(6)\qquad \tau(x)-\lambda^2\int_x^{b_j}(y-x)\tau(y)dy=f(x),\qquad (x\in l_j=(a_j,b_j))$$

where we have set $\lambda^2=B$. Equation (6) is easily solved, giving

$$(7)\qquad \tau(x)=f(x)+\lambda\int_x^{b_j}f(y)\sinh\lambda(y-x)dy.$$

Thus, we have proved that the solution $\tau(x)$ of eq. (1), and of eq. (3) as well, must be represented by (7) in terms of $f(x)$. However, the converse is not necessarily true. In order that (7) gives a solution of (3), or of (1), constants P_{jk} and q_m ($j=1,2,\cdots,\nu$: $k=1,2$: $m=0,1,\cdots,\nu+1$) contained in $f(x)$ must be chosen suitably. In a following section, a system of simultaneous linear equations with respect to the P's and q's will be derived, which will be a necessary and sufficient condition for (7) to give a solution of (3), or of (1).

3. To begin with, theorems which play an important role in the following calculations will be stated.

Theorem 1 (Hardy-Poincaré-Bertrand [2]).

$$(8)\qquad \int_l\frac{dy}{y-x}\int_l\frac{\phi(y,\zeta)}{\zeta-y}d\zeta=-\pi^2\phi(x,x)+\int_l d\zeta\int_l\frac{\phi(y,\zeta)}{(\zeta-y)(y-x)}dy.$$

Theorem 2 (Hayashi [1]).

$$(9)\qquad H_n(x)\equiv\frac{1}{\pi i}\int_l\frac{y^nX(y)}{y-x}dy=\begin{cases}-\sum_{m=0}^{n-\nu}\beta_{m-n}x^m, & (\nu\le n)\\ 0, & (0\le n\le\nu-1)\end{cases}$$

where

$$X(z)=1\Big/\sqrt{\prod_{j=1}^{\nu}(z-a_j)(z-b_j)}=\sum_{m=-\infty}^{-1}\beta_m z^m,\qquad |z|\gg 1.$$

Corollary (Hayashi [1]).

$$\int_l \frac{X(y)}{(\zeta-y)(y-x)}dy=0 \qquad (x, \zeta \in l, x \neq \zeta). \tag{10}$$

For the proof of these theorems, the reader is referred to [1] and [2].

On substituting (7) in (3), and changing the order of integration, we have

$$\begin{aligned}\pi i h(x)-\int_l \left\{\frac{1}{y-x}-A(y-x)-B(y-x)ln|x-y|\right\}f(y)dy \\ =\lambda \sum_{j=1}^{\nu}\int_{a_j}^{b_j} f(\zeta)d\zeta \int_{a_j}^{\zeta}\left\{\frac{1}{y-x}-A(y-x)\right. \\ \left.-B(y-x)ln|x-y|\right\} \sinh \lambda(\zeta-y)dy\end{aligned} \tag{11}$$

where the inner integral on the right hand can be calculated to be

$$\begin{aligned}&\int_{a_j}^{\zeta}\left\{\frac{1}{y-x}-A(y-x)-B(y-x)ln|x-y|\right\} \sinh \lambda(\zeta-y)dy \\ &=\lambda\left\{(\zeta-x)ln|\zeta-x|+\frac{A}{\lambda^2}(\zeta-x)\right\}-\left\{ln|a_j-x|+\frac{A+\lambda^2}{\lambda^2}\right\} \sinh \lambda(\zeta-a_j) \\ &\quad -\lambda\left\{(a_j-x)ln|a_j-x|+\frac{A}{\lambda^2}(a_j-x)\right\} \cosh \lambda(\zeta-a_j)\end{aligned}$$

which reduces (11) to

$$\begin{aligned}\pi i h(x)-\int_l \frac{f(y)}{y-x}dy \\ =-\lambda \sum_{j=1}^{\nu}\left[\left\{ln|a_j-x|+\frac{A+\lambda^2}{\lambda^2}\right\}\int_{a_j}^{b_j} f(y) \sinh \lambda(y-a_j)dy\right. \\ \left.+\lambda\left\{(a_j-x)ln|a_j-x|+\frac{A}{\lambda^2}(a_j-x)\right\}\int_{a_j}^{b_j} f(y) \cosh \lambda(y-a_j)dy\right].\end{aligned} \tag{12}$$

On substituting (5) in (12) and making use of (8), (9), and (10), it can be proved that the left and right hand sides of (12) are rewritten, respectively, as follows,

$$\text{L.H.S.}=-i\pi\sum_{j=1}^{\nu}(P_{j0}+P_{j1}x)ln|a_j-x|-i\pi\sum_{m=\nu}^{\nu+1} q_m H_m(x). \tag{13}$$

$$\begin{aligned}\text{R.H.S.}=-\sum_{j=1}^{\nu}\left[(A_\lambda h_s^j+Aa_j h_c^j)+\sum_{k=0}^{1}\sum_{i=1}^{\nu}P_{ik}(A_\lambda S^{ijk}+Aa_j C^{ijk})\right. \\ \left.+\sum_{m=0}^{\nu+1} q_m(A_\lambda S_m^j+Aa_j C_m^j)\right]+Ax\sum_{j=1}^{\nu}\left[h_c^j+\sum_{k=0}^{1}\sum_{i=1}^{\nu}P_{ik}C^{ijk}+\sum_{m=0}^{\nu+1} q_m C_m^j\right] \\ -\sum_{j=1}^{\nu} ln|a_j-x|\left[\lambda h_s^j+Ba_j h_c^j+\sum_{k=0}^{1}\sum_{i=1}^{\nu}P_{ik}(\lambda S^{ijk}+Ba_j C^{ijk})\right. \\ \left.+\sum_{m=0}^{\nu+1} q_m(\lambda S_m^j+Ba_j C_m^j)\right] \\ +B\sum_{j=1}^{\nu} x\, ln|a_j-x|\cdot\left[h_c^j+\sum_{k=0}^{1}\sum_{i=1}^{\nu}P_{ik}\, C^{ijk}+\sum_{m=0}^{\nu+1} q_m C_m^j\right]\end{aligned} \tag{14}$$

where we have set

$$\left.\begin{matrix}S^j(\zeta)\\ C^j(\zeta)\end{matrix}\right\}=\frac{1}{\pi i}\int_{l_j}\frac{X(y)}{\zeta-y}\left\{\begin{matrix}\sinh \lambda(y-a_j)\\ \cosh \lambda(y-a_j)\end{matrix}\right\}dy \tag{15}$$

$$\left.\begin{matrix} h_s^j \\ h_c^j \end{matrix}\right\} = \int_l \frac{h(\zeta)}{X(\zeta)} \left\{\begin{matrix} S^j(\zeta) \\ C^j(\zeta) \end{matrix}\right\} d\zeta, \qquad \left.\begin{matrix} S^{ijk} \\ C^{ijk} \end{matrix}\right\} = \int_l \frac{\zeta^k ln\,|a_j-\zeta|}{X(\zeta)} \left\{\begin{matrix} S^j(\zeta) \\ C^j(\zeta) \end{matrix}\right\} d\zeta$$

$$\left.\begin{matrix} S_m^j \\ C_m^j \end{matrix}\right\} = \int_{l_j} y^m X(y) \left\{\begin{matrix} \sinh \lambda(y-a_j) \\ \cosh \lambda(y-a_j) \end{matrix}\right\} dy, \qquad A_\lambda = \frac{A+\lambda^2}{\lambda}.$$

The necessary and sufficient condition for (7) to satisfy (3) is that (13) and (14) are equal to each other identically with respect to x. Because of the independent property of $1, x, ln\,|a_j-x|$ and $xln\,|a_j-x|$, this condition is equivalent to the following system of simultaneous linear equations with respect to P_{jk} and q_m ($j=1,2,\cdots,\nu: k=1,2: m=0,1,\cdots,\nu+1$).

$$\text{(16)} \quad i\pi\beta_{-\nu}q_\nu + i\pi\beta_{-\nu-1}q_{\nu+1} + \sum_{k=0}^{1}\sum_{i=1}^{\nu} P_{ik} \sum_{j=1}^{\nu} (A_\lambda S^{ijk} + Aa_j C^{ijk})$$
$$+\sum_{m=0}^{\nu+1} q_m \sum_{j=1}^{\nu} (A_\lambda S_m^j + Aa_j C_m^j) = -\sum_{j=1}^{\nu} (A_\lambda h_s^j + Aa_j h_c^j)$$
$$\frac{i\pi}{A}\beta_{-\nu}q_{\nu+1} - \sum_{k=0}^{1}\sum_{i=1}^{\nu} P_{ik} \sum_{j=1}^{\nu} C^{ijk} - \sum_{m=0}^{\nu+1} q_m \sum_{j=1}^{\nu} C_m^j = \sum_{j=1}^{\nu} h_c^j$$
$$\times i\pi P_{j0} - \sum_{k=0}^{1}\sum_{i=1}^{\nu} P_{ik}(\lambda S^{ijk} + Ba_j C^{ijk}) - \sum_{m=0}^{\nu+1} q_m(\lambda S_m^j + Ba_j C_m^j) = (\lambda h_s^j + Ba_j h_c^j)$$
$$\frac{i\pi}{B}P_{j1} + \sum_{k=0}^{1}\sum_{i=1}^{\nu} P_{ik} C^{ijk} + \sum_{m=0}^{\nu+1} q_m C_m^j = -h_c^j.$$

As a conclusion, a function $\tau(x)$ defined by (7) in terms of the P's and q's which satisfy (16) gives a solution of (3). Since (16) is composed of $2\nu+2$ equations with respect to $3\nu+2$ unknowns, a solution of (3) contains ν undetermined parameters in general, in accordance with the known result [2].

4. Let the kernel of (1) be denoted by $\psi_0(x, y)$ and let $\tau(x)$ be a solution of (3) obtained above. If $\tau(x)$ is substituted,

$$\int_l \psi_0(x, y)\tau(y)dy - g(x)$$

may not necessarily be zero, but must be equal to some constant, since

$$\frac{d}{dx}\left\{\int_l \psi_0(x, y)\tau(y)\,dy - g(x)\right\} = 0$$

is (3) and $\tau(x)$ is a solution of it. This implies that, in order that (7) gives a solution for (1), the ν remaining parameters should be determined by the further conditions

$$\text{(17)} \qquad \int_l \psi_0(x_j, y)\tau(y)dy - g(x_j) = 0 \qquad (j=1,2,\cdots,\nu)$$

where x_j is arbitrarily fixed point on l_j. Equation (17) represents ν simultaneous linear equations with respect to the P's and q's, whose explicit representation is straightforward. Equation (17) together with (16) comprise $3\nu+2$ equations with respect to $3\nu+2$ unknown parameters P and q, whose solution, when substituted in (7), gives the solution of (1).

5. We have shown that a solution of (3), or of (1), if it exists,

must be given by (7), with the constants P's and q's satisfying (16), or (16) and (17), and that the converse is also true. Thus, the existence and the uniqueness questions on a solution of (3), or of (1), have been reduced to those on solutions of (16), or (16) and (17), however, we shall not go further, because these questions depend on the choice of coefficients of the original equations, the ranges of integration, and the wave number k. We remark that the results obtained here are of interest in the analysis of acoustic and electromagnetic diffractions by a union of line segments of arbitrary width. Detailed discussions of these questions will be left to more specific, practical cases.

References

[1] Hayashi, Y.: The Dirichlet problem for the two-dimensional Helmholtz equation for an open boundary. J. Math. Anal. Appl., **44**, 489–530 (1973).

[2] Muskhelishvilli, N. I.: Singular Integral Equations. P. Noordhoff N.V., Groningen, Holland (1953).

[3] Gakhov, F. D.: Boundary Value Problems. Pergamon Press, Oxford (1966).