50. The Asymptotics of the Potential Functions of One-Sided Stable Processes

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1. Introduction. Let $x(t)$ be a temporally homogeneous independent increments process with only negative jumps, whose cumulant is

$$K(s)=\frac{1}{t}\log \boldsymbol{E}e^{sx(t)}=as+\frac{b}{2}s^2+\int_{-\infty}^{0}\left(e^{sx}-1-\frac{sx}{1+x^2}\right)\Pi(dx),$$

where $s\geqq 0$, $b\geqq 0$ and the Lévy measure Π is a measure which makes the above integral converge. We define $\zeta=\inf\{t\,;\,x(t)\leqq 0\}$ and $x^0(t)$, $t\in[0,\zeta)$, is the process obtained by killing $x(t)$ at the moment ζ. It is well known that $x^0(t)$ is a Markov process and the resolvent $\boldsymbol{R}_\lambda^0$ of the process $x^0(t)$ is given by

$$\boldsymbol{R}_\lambda^0 f(x)=\boldsymbol{E}_x\int_0^{\zeta}e^{-\lambda t}f(x(t))dt$$

for $\lambda>0$ and bounded measurable function $f(x)$. Here $\boldsymbol{E}_x$ and $\boldsymbol{P}_x$ are respectively conditional expectation and conditional probability under the condition $x(0)=x$.

In [3] it was proved for $\lambda>0$, $x>0$

$$\boldsymbol{R}_\lambda^0 f(x)=R_\lambda(x)\int_0^\infty e^{-\rho(\lambda)y}f(y)dy-\int_0^x R_\lambda(x-y)f(y)dy,$$

where $\rho(\lambda)$ is a solution of $K(s)=\lambda$, and the Laplace transform of $R_\lambda(x)$ is

$$\int_0^\infty e^{-sx}R_\lambda(x)dx=\frac{1}{K(s)-\lambda}\qquad \text{for } s>\rho(\lambda).$$

We call $R_\lambda(x)$ *resolvent function* and it was shown there exists $R(x)=\lim_{\lambda\to 0}R_\lambda(x)$, which we call *potential function*. If we put $\rho=\lim_{\lambda\to 0}\rho(\lambda)$, it is obvious

$$\int_0^\infty e^{-sx}R(x)dx=\frac{1}{K(s)}\qquad \text{for } s>\rho.$$

As application of this result [2], we obtain the formulas

$$\boldsymbol{E}_x(\zeta)=\frac{R(x)}{\rho}-\int_0^x R(y)dy,$$

$$\begin{aligned}&\boldsymbol{P}_x(x(\zeta)<z,\zeta<\infty)\\&\quad=R(x)\int_0^\infty e^{-\rho y}\Pi(-\infty,z-y)dy-\int_0^x R(x-y)\Pi(-\infty,z-y)dy,\end{aligned}$$

where $\Pi(-\infty,z-y)=\int_{-\infty}^{z-y}\Pi(du)$.

In the present note we give the asymptotics of $R(x)$, $\boldsymbol{E}_x(\zeta)$ and $\boldsymbol{P}_x(x(\zeta)<z)$ when $x\to\infty$ especially for one-sided stable processes with only negative jumps; $b=0$, $\Pi(du)=du/|u|^{\alpha+1}$ $(u<0)$. They are all Laplace transformed and Tauberian theorems [1] are applied for the proofs. Although some of these need certain tricks, but we omit here the details.

Remark. In [3] it is not investigated the case when $K(s)\leqq 0$ for all $s\geqq 0$. But investigation analogous to [3] makes us convince

$$\boldsymbol{R}_\lambda^0 f(x)=-\int_0^x R_\lambda(x-y)f(y)dy \text{ and } \int_0^\infty e^{-sx}R_\lambda(x)dy=\frac{1}{K(s)-\lambda} \text{ for } s>0,$$

$$\boldsymbol{E}_x(\zeta)=-\int_0^x R(y)dy,$$

$$\boldsymbol{P}_x(x(\zeta)<z,\zeta<\infty)=-\int_0^x R(x-y)\Pi(-\infty,z-y)dy.$$

These formulas are used in the following case $0<\alpha<1$, $a\leqq 0$.

2. The assymptotics when $x\to\infty$. The Lévy measure $\Pi(dx)=\dfrac{1}{|x|^{1+\alpha}}dx$.

1. $K(s)=as-\dfrac{\Gamma(1-\alpha)}{\alpha}s^\alpha$ $(0<\alpha<1)$

(i) $a>0$ $\quad R(x)\sim\dfrac{e^{\rho x}}{K'(\rho)}-\dfrac{\alpha\sin\alpha\pi}{\pi}x^{\alpha-1}, \qquad \rho=\left\{\dfrac{\Gamma(1-\alpha)}{a\alpha}\right\}^{1/(1-\alpha)},$

$a=0$ $\quad R(x)=-\dfrac{\alpha\sin\alpha\pi}{\pi}x^{\alpha-1},$

$a<0$ $\quad R(x)\sim-\dfrac{\alpha\sin\alpha\pi}{\pi}x^{\alpha-1}-a\left\{\dfrac{\alpha}{\Gamma(1-\alpha)}\right\}^2\dfrac{1}{\Gamma(2\alpha-1)}x^{2(\alpha-1)},$

(ii) $a\neq 0$ $\quad \boldsymbol{E}_x(\zeta)\sim\dfrac{\sin\alpha\pi}{\pi}x^\alpha+a\left\{\dfrac{\alpha}{\Gamma(1-\alpha)}\right\}^2\dfrac{1}{\Gamma(2\alpha)}x^{2\alpha-1},$

$a=0$ $\quad \boldsymbol{E}_x(\zeta)=\dfrac{\sin\alpha\pi}{\pi}x^\alpha,$

(iii) for every a $\quad \boldsymbol{P}_x(x(\zeta)<z)\sim 1-\alpha\dfrac{|z|}{x}.$

2. $K(s)=as+s\log s$ (one-sided Cauchy process)

(i) $R(x)\sim\dfrac{e^{\rho x}}{K'(\rho)}-\dfrac{1}{\log x}, \qquad \rho=e^{-a},$

(ii) $\boldsymbol{E}_x(\zeta)\sim\dfrac{x}{\log x},$

(iii) $\boldsymbol{P}_x(x(\zeta)<z)\sim 1-\dfrac{1}{\log x}\left\{\log|z|+\gamma+e^{|z|\rho}\int_{|z|\rho}^\infty\dfrac{e^{-t}}{t}dt\right\}$

$$\sim 1-\frac{\log|z|}{\log x} \text{ for sufficiently large } |z|$$

(γ: Euler constant).

3. $K(s)=as-\dfrac{\Gamma(1-\alpha)}{\alpha}s^{\alpha}\ (1<\alpha<2)$

(i) $a>0 \quad R(x)\sim\dfrac{1}{a}+\dfrac{1}{a^2\alpha(1-\alpha)}x^{1-\alpha},$

$a=0 \quad R(x)=-\dfrac{\alpha\sin\alpha\pi}{\pi}x^{\alpha-1},$

$a<0 \quad R(x)\sim\dfrac{e^{\rho x}}{K'(\rho)}+\dfrac{1}{a}, \qquad \rho=\left\{\dfrac{a\alpha}{\Gamma(1-\alpha)}\right\}^{1/(\alpha-1)},$

(ii) $a\geqq 0 \quad \boldsymbol{E}_x(\zeta)=\infty,$

$a<0 \quad \boldsymbol{E}_x(\zeta)\sim-\dfrac{x}{a},$

(iii) $a>0 \quad \boldsymbol{P}_x(x(\zeta)<z)\sim\dfrac{1}{a\alpha(\alpha-1)x^{\alpha-1}}\left\{1-(\alpha-1)\dfrac{|z|}{x}\right\},$

$a=0 \quad \boldsymbol{P}_x(x(\zeta)<z)\sim 1-(\alpha-1)\dfrac{|z|}{x},$

$a<0 \quad \boldsymbol{P}_x(x(\zeta)<z)\sim\dfrac{c(z)}{a}-\dfrac{1}{a\alpha(\alpha-1)\,|z|^{\alpha-1}},$

$$=\frac{1}{a}\int_0^{\infty}(e^{-\rho y}-1)\Pi(-\infty,z-y)dy,$$

where $c(z)=\dfrac{1}{\alpha}\displaystyle\int_0^{\infty}e^{-\rho y}\frac{1}{|y-z|^{\alpha}}dy.$

References

[1] W. Feller: An introduction to probability theory and its application. vol. 2, John Wiley, New York (1966).

[2] В. С. Королюк, В. Н. Супрун, и В. М. Шуренков: Метод потенциала в граничных задачах для процессов с независимыми приращениями и скачками одного знака. Теория вероят. и ее примен., **21**, 253–259 (1976).

[3] В. Н. Супрун и В. М. Шуренков: О резольвенте процесса с независимыми приращениями, обрывающегося в момент выхода на отрицательную полуось. «Исследования по теории случайных процессов», Изд-во Института математики АН УССР, 170–174 (1976).