49. Monodromy Preserving Deformation and Its Application to Soliton Theory. II

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§ 1. **Introduction.** This is a sequel of the preceding papers [1], [2]. In the previous article [2], the author showed that the multi-soliton solutions of the sine-Gordon equation are governed by the iso-monodromic deformation equations. The purpose of the present note is to extend the result in [2] to the Pohlmeyer and Lund-Regge system (PLR) [3], [4]

$$(1.1)\qquad u_{\xi\eta}-\frac{v_\xi v_\eta \sin(u/2)}{2\cos^3(u/2)}+\sin u=0,$$

$$v_{\xi\eta}+\frac{u_\xi v_\eta+u_\eta v_\xi}{\sin u}=0$$

and the non-linear Schrödinger equation (NLS)

$$(1.2)\qquad u_\eta-iu_{\xi\xi}-2i|u^2|u=0.$$

The multi-soliton solutions of these equations are related to the monodromy preserving deformations of the following 2×2 first order systems, respectively:

$$(1.3)\qquad PY=0,\qquad P=\frac{d}{dx}-\Big(G+Fx^{-1}+Ex^{-2}+\sum_{j=1}^{N}\frac{H_j}{x-a_j}\Big),$$

$$(1.4)\qquad PY=0,\qquad P=\frac{d}{dx}-\Big(Gx+F+\sum_{j=1}^{N}\frac{H_j}{x-a_j}\Big).$$

The reader is referred to the previous paper [2], in which the deformation theory for the above equations was developed.

Another purpose of the present note is to investigate the Hamiltonian structure of the deformation equations for the above systems (1.3) and (1.4), and to calculate explicitly the "τ-function" in the case of PLR and NLS (cf. [8], [9], [10]). It is known that these "τ-function" are deeply connected with the Fredholm determinant of Gelfand-Levitan-Marchenko equation linearlizing PLR and NLS (cf. [10]).

§ 2. **Application to PLR and NLS.** PLR (1.1) is equivalent to the compatibility condition of the system of differential equations (cf. [3], [4])

$$(2.1)\qquad \Big(\frac{\partial}{\partial\xi}-i\begin{bmatrix} & -a^* \\ -a & \end{bmatrix}-ix/2\begin{bmatrix}1 & \\ & -1\end{bmatrix}\Big)Y=0,$$

$$\left(\frac{\partial}{\partial \eta}-ix^{-1}/2\begin{bmatrix}\cos u & \exp(-i\omega)\sin u \\ \exp(i\omega)\sin u & -\cos u\end{bmatrix}\right)Y=0,$$

where $x \in \boldsymbol{C}$, $(\xi, \eta) \in \boldsymbol{R}^2$, $a=i\exp(i\omega)\sin u/2\cos u$,

$$\omega_\xi=v_\xi\cos u/2\cos^2(u/2), \qquad \omega_\eta=v_\eta/2\cos^2(u/2),$$

and $*$ denotes the complex conjugate. Also NLS (1.2) is the compatibility condition of

$$\left(\frac{\partial}{\partial \xi}-ix\begin{bmatrix}-1 & \\ & 1\end{bmatrix}-\begin{bmatrix} & -u \\ u^* & \end{bmatrix}\right)Y=0, \tag{2.2}$$

$$\left(\frac{\partial}{\partial \eta}-x^2\begin{bmatrix}-2i & \\ & 2i\end{bmatrix}-x\begin{bmatrix} & -2u \\ 2u^* & \end{bmatrix}-\begin{bmatrix}i|u|^2 & -iu_\xi \\ -iu^* & -i|u|^2\end{bmatrix}\right)Y=0,$$

where $x \in \boldsymbol{C}$, $(\xi, \eta) \in \boldsymbol{R}^2$ (cf. [7]).

Let us review Date's direct construction method of N-soliton solutions of PLR and NLS (cf. [5], [6]). First we construct a matrix solution of (2.1)–(2.2) satisfying the following conditions:

$$Y(x, \xi, \eta)=\hat{Y}(x, \xi, \eta)x^N\begin{bmatrix}e^\theta & \\ & e^{-\theta}\end{bmatrix} \tag{2.3}$$

where

$$\hat{Y}(x, \xi, \eta)=I+\sum_{j=1}^{N} Y_j(\xi, \eta)x^{-j}, \qquad Y_j=\begin{bmatrix}y_{1,N-j} & y^*_{2,N-j} \\ -y_{2,N-j} & y^*_{1,N-j}\end{bmatrix}$$

and $\theta=i/2(\xi x+\eta x^{-1})$ in the case of (2.1), while $\theta=-i(2\eta x^2+\xi x)$ in the case of (2.2). Furthermore $Y(x, \xi, \eta)$ is assumed to satisfy the degenerating conditions

$$Y(\alpha_j, \xi, \eta)\begin{bmatrix}1 \\ -c_j\end{bmatrix}=0, \quad Y(\alpha^*_j, \xi, \eta)\begin{bmatrix}c^*_j \\ 1\end{bmatrix}=0, \quad (j=1, \cdots, N), \tag{2.4}$$

where α_j $(j=1, \cdots, N)$ are complex constants such that $\alpha_j \neq \alpha_k$ for $j \neq k$, $\alpha_j \neq \alpha^*_k$ for any j, k, and c_j $(j=1, \cdots, N)$ are non-zero complex constants. This condition is common to both cases. We define a $2N \times 2N$ matrix W by

$$W=(a_0, \cdots, a_{N-1}, a'_0, \cdots, a'_{N-1}) \tag{2.5}$$

where

$$a_l={}^t(\alpha_1^l e(\alpha_1), \cdots, \alpha_N^l e(\alpha_N), c^*_1\alpha^{*l}_1 e(\alpha^*_1), \cdots, c^*_N\alpha^{*l}_N e(\alpha^*_N)),$$

$$a'_l={}^t(-c_1\alpha_1^l e(\alpha_1)^{-1}, \cdots, -c_N\alpha_N^l e(\alpha_N)^{-1}, \alpha^{*l}_1 e(\alpha^*_1)^{-1}, \cdots, \alpha^{*l}_N e(\alpha^*_N)^{-1}),$$

and $e(\alpha)=\exp(i/2(\xi\alpha+\eta\alpha^{-1}))$ in the case of (2.1), while $e(\alpha)=\exp(-2i\eta\alpha^2-i\xi\alpha)$ in the case of (2.2), for $\alpha=\alpha_j$ or α^*_j $(j=1, \cdots, N)$. If det W does not identically vanish, $Y(x, \xi, \eta)$ is uniquely determined by the above conditions.

Proposition (Date [6]). *Case* (2.1). *$Y(x, \xi, \eta)$ solves the equation*

$$dY=\Omega Y, \tag{2.6}$$

where

$$\Omega=\left(ix/2\begin{bmatrix}1 & \\ & -1\end{bmatrix}+i\begin{bmatrix} & -y^*_{2,N-1} \\ -y_{2,N-1} & \end{bmatrix}\right)d\xi$$

$$+ix^{-1}/2(|y_{1,0}|^2+|y_{2,0}|^2)^{-1}\begin{bmatrix}|y_{1,0}|^2-|y_{2,0}|^2 & -2y_{1,0}y^*_{2,0} \\ -2y^*_{1,0}y_{2,0} & |y_{2,0}|^2-|y_{1,0}|^2\end{bmatrix}d\eta.$$

Hence $Y(x,\xi,\eta)$ *is a solution of* (2.1) *by the identification*

$$a=y_{2,N-1},\qquad \cos u=(|y_{1,0}|^2-|y_{2,0}|^2)(|y_{1,0}|^2+|y_{2,0}|^2)^{-1},$$
$$\exp(i\omega)\sin u=-2y^*_{1,0}y_{2,0}(|y_{1,0}|^2+|y_{2,0}|^2)^{-1}.$$

The pair of functions

$$u=\arccos\{(|y_{1,0}|^2-|y_{2,0}|^2)(|y_{1,0}|^2+|y_{2,0}|^2)^{-1}\},$$
$$v=-i\log(y_{2,0}y^{*-1}_{2,0})+v_0(v_0\in \boldsymbol{R})$$

is an N-soliton solution of PLR (1.1). *Case* (2.2). $Y(x,\xi,\eta)$ *satisfies the equation*

$$dY=\Omega Y, \tag{2.7}$$

where

$$\begin{aligned}\Omega=&\left(\mathrm{ix}\begin{bmatrix}-1 & \\ & 1\end{bmatrix}+\begin{bmatrix} & 2iy^*_{2,N-1}\\ 2iy_{2,N-1} & \end{bmatrix}\right)d\xi\\ &+\left(x^2\begin{bmatrix}-2i & \\ & 2i\end{bmatrix}+x\begin{bmatrix} & 4iy^*_{2,N-1}\\ 4iy_{2,N-1} & \end{bmatrix}\right.\\ &\left.+\begin{bmatrix}4i\,|y_{2,N-1}|^2 & 4iy^*_{2,N-1}-4iy^*_{1,N-1}y^*_{2,N-1}\\ 4iy_{2,N-1}-4iy_{1,N-1}y_{2,N-1} & 4i\,|y_{2,N-1}|^2\end{bmatrix}\right)d\eta.\end{aligned}$$

Therefore $Y(x,\xi,\eta)$ *is a solution of* (2.2) *by the identification*

$$u=-2iy^*_{2,N-1},\qquad u_\xi=-4y^*_{2,N-1}+4y^*_{1,N-1}y_{2,N-1}.$$

Then $u=-2iy^*_{2,N-1}$ *is an N-soliton solution of* NLS (1.2).

Next we search for the x-equations satisfied by $Y(x,\xi,\eta)$. After a little computation, $Y(x,\xi,\eta)$ is shown to solve the following equation:

Case (2.1).

$$\frac{\partial Y}{\partial x}=\left\{x^{-2}E+x^{-1}F+G+\sum_{j=1}^{N}\left(\frac{H_{\alpha_j}}{x-\alpha_j}+\frac{H_{\alpha_j^*}}{x-\alpha_j^*}\right)\right\}Y \tag{2.8}$$

where

$$G=i\xi/2\begin{bmatrix}1 & \\ & -1\end{bmatrix},\qquad E=K\tilde{E}K^{-1},$$
$$\tilde{E}=-i\eta/2\begin{bmatrix}1 & \\ & -1\end{bmatrix},\qquad K=\begin{bmatrix}y_{1,0} & y^*_{2,0}\\ -y_{2,0} & y^*_{1,0}\end{bmatrix}$$
$$F+\sum_{j=1}^{N}(H_{\alpha_j}+H_{\alpha_j^*})=\begin{bmatrix}N & -i\xi y_{2,N-1}\\ -i\xi y_{2,N-1} & N\end{bmatrix},$$

Case (2.2).

$$\frac{\partial Y}{\partial x}=\left\{Gx+F+\sum_{j=1}^{N}\left(\frac{H_{\alpha_j}}{x-\alpha_j}+\frac{H_{\alpha_j^*}}{x+\alpha_j^*}\right)\right\}Y \tag{2.9}$$

where

$$G=\begin{bmatrix}-4i\eta & \\ & 4i\eta\end{bmatrix},$$
$$F=\begin{bmatrix}-i\xi & 8i\eta y^*_{2,N-1}\\ 8i\eta y_{2,N-1} & i\xi\end{bmatrix}$$
$$\sum_{j=1}^{N}(H_{\alpha_j}+H_{\alpha_j^*})=(h_{\mu\nu})$$
$$h_{11}=N+8i\eta\,|y_{2,N-1}|^2,\qquad h_{22}=h^*_{11}$$

$$h_{21}=8i\eta y_{2,N-2}-8i\eta y_{1,N-1}y_{2,N-1}+2i\xi y_{2,N-1}, \qquad h_{22}=-h_{21}^{*}.$$

We note that the eigenvalues of H_α ($\alpha=\alpha_j, \alpha_j^*$, $j=1,\cdots,N$) are 0 and 1 in both cases. Next we determine the global connection structure for the solution of (2.9). The Stokes multipliers and the formal monodromy of (2.9) around the infinity are all trivial. We introduce invertible matrices T_α, Q_α ($\alpha=\alpha_j, \alpha_j^*$, $j=1,\cdots,N$) as follows:

$$\begin{aligned} &H_\alpha=T_\alpha \operatorname{diag}(0,1)T_\alpha^{-1} \\ &Y(x,\xi,\eta)=T_\alpha Y_\alpha(x,\xi,\eta)Q_\alpha. \end{aligned} \tag{2.10}$$

Here Y_α is the normalized solution of (2.9) at $x=\alpha$ expressed as

$$Y_\alpha=(x-\alpha)^J\Phi_\alpha(x)(x-\alpha)^{\left[\begin{smallmatrix}0 & \\ l_\alpha & 0\end{smallmatrix}\right]},$$

where $J=\operatorname{diag}(0,1)$, and $\Phi_\alpha(x)$ is holomorphic near $x=\alpha$, and $\Phi_\alpha(\alpha)=I$. In the present case, $l_\alpha=0$, because logalithmic terms are absent in $Y(x,\xi,\eta)$. By choosing an appropriate T_α, it is shown that

$$Q_\alpha=\begin{bmatrix}1 & c\\ & 1\end{bmatrix},$$

where $c=c_j$ for $\alpha=\alpha_j$, and $c=-c_j^{*-1}$ for $\alpha=\alpha_j^*$. As we have seen above, the deformation properties in the sense of [2] hold. In a parallel argument as above, we know that the equation (2.8) is deformed with keeping the deformation properties. Hence $Y(x,\xi,\eta)$ should satisfy the equation $dY=\tilde{\Omega}Y$. The rational 1-form $\tilde{\Omega}$ is determined by the formula in Theorems 1 and 2 of [2], and coincides with Ω in (2.6)–(2.7), respectively.

Summing up, we have our main result.

Theorem 1. *F, K, and H_α ($\alpha=\alpha_j, \alpha_j^*$, $j=1,\cdots,N$) satisfy the deformation equations in the sense of* [2] (*refer to* (3.1)–(3.2) *in* §3, *where H_j are replaced by H_α*). *These equations characterize the N-soliton solutions of* PLR *and* NLS, *respectively.*

§3. Hamiltonian structure and τ-function. In this paragraph, we will describe the Hamiltonian structure of the following two types of completely integrable system

$$\begin{aligned} &dK=K\{d\tilde{E}, K^{-1}FK\}_{\tilde{E}}+\Big\{dG, F+\sum_{j=1}^{N}H_j\Big\}_G K \\ &dF=[\Phi,E]+[\Theta,G]+[\Psi,F]-\sum_{j=1}^{N}a_j^{-1}[\Theta,H_j] \\ &dH_j=[a_j\Phi+\Psi+a_j^{-1}\Theta,H_j], \qquad j=1,\cdots,N, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &dF=\Psi+[\Theta,F]+\sum_{j=1}^{N}a_j[\Phi,H_j]+\sum_{j=1}^{N}[\Psi,H_j] \\ &dH_j=[a_j^2\Phi+a_j\Psi+\Theta,H_j], \qquad j=1,\cdots,N. \end{aligned} \tag{3.2}$$

Here $F=(F_{\mu\nu})$, $H_j=(H_{j,\mu\nu})$ ($j=1,\cdots,N$), $K=(K_{\mu\nu})$, and $E=K\tilde{E}K^{-1}$ are 2×2 matrices, $G=\operatorname{diag}(g_1,g_2)$, $\tilde{E}=\operatorname{diag}(e_1,e_2)$ with $g_1\neq g_2$, and $e_1\neq e_2$, and a_j ($j=1,\cdots,N$) are mutually distinct constants. In case (3.1), d denotes the exterior differentiation with respect to g_μ and e_μ ($\mu=1,2$), and Φ, Ψ, and Θ are given by

(3.3) $\quad \Phi = dG \quad \Psi = \left\{dG, F + \sum_{j=1}^{N} H_j\right\}_G, \quad \Theta = -Kd\tilde{E}K^{-1}.$

In case (3.2), d denotes the exterior differentiation with respect to $g_1, g_2, f_1 = F_{11}$ and $f_2 = F_{22}$, and Φ, Ψ, Θ are defined by

(3.4) $\quad \Phi = \frac{1}{2}dG, \quad \Psi = dF^{(+)} + \{\Phi, F\}_G, \quad F^{(+)} = \mathrm{diag}\,(f_1, f_2),$

$$\Theta = \left\{\Phi, \sum_{j=1}^{N} H_j\right\}_G + \{\Psi, F\}_G + \frac{1}{2}\,\mathrm{diag}\left(F_{12}F_{21}d\left(\frac{1}{g_1 - g_2}\right), F_{21}F_{12}d\left(\frac{1}{g_2 - g_1}\right)\right).$$

The bracket notation $\{\ ,\ \}$ was introduce in [1].

Let $Y = \hat{Y}x^{D_\infty}\exp(xG)$, where

$$\hat{Y} = I + \sum_{l=1}^{\infty} Y_l x^{-l}, \quad \text{and} \quad Z = \hat{Z}x^{D_0}\exp(-x^{-1}\tilde{E}),$$

where

$$\hat{Z} = I + \sum_{l=1}^{\infty} Z_l x^l,$$

be the normalized formal solution matrix of (1.3) at the infinity and the origin, respectively. Likewise let

$$Y = \hat{Y}x^D \exp(1/2x^2G + xF^{(+)}), \quad \text{where} \quad \hat{Y} = I + \sum_{l=1}^{\infty} Y_l x^{-l},$$

be the normalized formal matrix solution of (1.4) at the infinity.

We give a description of the Hamiltonian structure for the systems (3.1), (3.2). That of (3.1) was suggested by T. Miwa.

Theorem 2. *We define the* 1-*form* ω

Case (3.1) $\quad \omega = \mathrm{trace}\, Z_1 d\tilde{E} - \mathrm{trace}\, Y_1 dG,$

Case (3.2) $\quad \omega = -\mathrm{trace}\left(Y_1 dF^{(+)} + Y_2 dG - \frac{1}{2}Y_1^2 dG\right).$

Here Y_1, Z_1 *and* Y_1, Y_2 *are the first or the second coefficients of the formal matrix solution of* (1.3)–(1.4), *respectively. We introduce the Poisson bracket* $\{\,,\}$ *among the dependent variable* F, K *and* H_j *through*

Case (3.1) $\quad \{(FK)_{\mu\nu}, (K^{-1})_{\mu'\nu'}\} = \delta_{\mu\nu'}\delta_{\nu\mu'},$

$\{H_{j,\mu\nu}, H_{k,\mu'\nu'}\} = \delta_{jk}(\delta_{\mu\nu'}H_{j,\mu'\nu} - \delta_{\mu'\nu}H_{k,\mu\nu'})$

all other combinations of $(FK)_{\mu\nu}, (K^{-1})_{\mu\nu}, H_{j,\mu\nu}$ *are zero.*

Case (3.2) $\quad \{F_{12}, F_{21}\} = 1,$

$\{H_{j,\mu\nu}, H_{k,\mu'\nu'}\} = \delta_{jk}(\delta_{\mu\nu'}H_{j,\mu'\nu} - \delta_{\mu'\nu}H_{k,\mu\nu'}),$

all other combinations of $F_{\mu\nu}, H_{j,\mu\nu}$ *are zero.*

Here $F_{\mu\nu} = (g_1 - g_2)^{1/2}\tilde{F}_{\mu\nu}$ $(\mu \neq \nu)$. *Then the systems* (3.1)–(3.2) *are written in a Hamiltonian system*

Case (3.1) $\quad dK = \{K, \omega\}, \qquad dF = \{F, \omega\}$

$dH_j = \{H_j, \omega\} \qquad (j = 1, \cdots, N),$

Case (3.2) $\quad d\tilde{F} = \{\tilde{F}, \omega\}, \qquad \tilde{F} = \begin{bmatrix} & \tilde{F}_{12} \\ \tilde{F}_{21} & \end{bmatrix}$

$dH_j = \{H_j, \omega\} \qquad (j = 1, \cdots, N),$

with the Hamiltonian 1-*form* ω *given above.*

For any solution of (3.1)–(3.2), ω is shown to be closed. If we write $\omega=\sum_{j=1}^{N} h_j dt_j$, where $t_1=g_1$, $t_2=g_2$, $t_3=e_1$, $t_4=e_2$, in Case (3.1) (resp. $t_3=f_1$, $t_4=f_2$ in Case (3.2)), we know that $\frac{\partial h_j}{\partial t_i}=\frac{\partial h_i}{\partial t_j}$ for any i, j. Since the 1-form ω is closed for each solution of (3.1)–(3.2), there exists a function τ, unique up to a constant multiple, satisfying $\omega=d\log\tau$. In our case, the solution F, K, and H_α in (2.8)–(2.9) are expressible in the terms of $y_{n,l}y^*_{n,l}$ and hence by α_j, α^*_j, and c_j, c^*_j $(i=1, \cdots, N)$, so that an explicit form of "τ-function" is derived. After a little computation, we obtain

$$(3.5)\qquad \begin{aligned}&(\mathrm{PLR})\quad \tau(\xi,\eta)=\mathrm{const}\, e^{\xi\eta/2}\det W,\\ &(\mathrm{NLS})\quad \tau(\xi,\eta)=\mathrm{const}\,\det W.\end{aligned}$$

Here W is given by (2.5). If we define an $N\times N$ matrix $C=(C_{\mu\nu})$

$$(3.6)\qquad C_{\mu\nu}=\frac{c_\mu}{\alpha_\mu-\alpha^*_\nu}\frac{g(\alpha_\mu)}{\dot g(\alpha^*_\nu)}e(\alpha_\mu)^{-1}e(\alpha^*_\nu),$$

where

$$g(x)=\prod_{\mu=1}^{\Lambda}(x-\alpha^*_\mu),\qquad \dot g=\frac{dg}{dx},\quad \text{and } e(\alpha)$$

is given in (2.5), we obtain the final expression for $\tau(\xi,\eta)$

$$(3.7)\qquad \begin{aligned}&(\mathrm{PLR})\quad \tau(\xi,\eta)=\mathrm{const}\, e^{\xi\eta/2}\prod_{j=1}^{N} e(\alpha_j)e(\alpha^*_j)^{-1}\det(I+C^*C)\\ &(\mathrm{NLS})\quad \tau(\xi,\eta)=\mathrm{const}\prod_{j=1}^{N} e(\alpha_j)e(\alpha^*_j)^{-1}\det(I+C^*C).\end{aligned}$$

Here C^* is the complex conjugate of C. Lastly we remark that, if C is pure imaginary i.e. $C^*=-C$ in (PLR) case, the "τ-function" reduces to that of sine-Gordon equation.

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