46. *The First Cohomology Groups of Infinite Dimensional Lie Algebras*

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Introduction. Let L be an infinite dimensional *formal* Lie algebra corresponding to some infinite transformation group. We are interested in the first cohomology group $H^1(L)$ of L with adjoint representation. In this paper we will treat the following two types of infinite dimensional Lie algebras;

(a) infinite dimensional *transitive flat* Lie algebras,

(b) infinite dimensional intransitive Lie algebras $L[W^*]$ whose transitive parts L are infinite and simple.

Throughout this paper, all vector spaces and Lie algebras are assumed to be defined over the field C of complex numbers.

1. Let V be a finite dimensional vector space. We denote by $D(V)$ the Lie algebra of all formal vector fields over V. The Lie algebra $D(V)$ can be written as $D(V)=\prod_{p\geqq 0} V\otimes S^p(V^*)$ (complete direct sum), where $S^p(V^*)$ denotes p-times symmetric tensor of the dual space V^* of V. By *a transitive flat Lie algebra* we mean a Lie subalgebra $L=\prod_{p\geqq -1}\mathfrak{g}_p$ of $D(V)$ satisfying the following conditions:

(1) Each $\mathfrak{g}_p$ is a subspace of $V\otimes S^{p+1}(V^*)$.

(2) $\mathfrak{g}_{-1}=V$ (transitivity condition).

Since L is a Lie algebra, it must hold that

(3) $[\mathfrak{g}_p, \mathfrak{g}_q]\subset\mathfrak{g}_{p+q}$.

A Lie subalgebra $\mathfrak{g}_0$ is called a linear isotropy algebra of L. We say that a Lie algebra $L=\prod_{p\geqq -1}\mathfrak{g}_p$ is *derived* from $\mathfrak{g}_0$ if each $\mathfrak{g}_p$ coincides with the p-th prolongation of $\mathfrak{g}_0$.

We now give two criteria for $H^1(L)$ to be of finite dimension.

Theorem 1. *Let $L=\prod_{p\geqq -1}\mathfrak{g}_p$ be an infinite transitive flat Lie algebra with a semi-simple linear isotropy algebra. Then $H^1(L)$ is finite dimensional.*

Theorem 2. *Let $L=\prod_{p\geqq -1}\mathfrak{g}_p$ be an infinite transitive flat Lie algebra whose linear isotropy algebra $\mathfrak{g}_0$ contains a trivial center. Then $H^1(L)$ is finite dimensional. Furthermore if L is derived from $\mathfrak{g}_0$, then $H^1(L)$ is isomorphic to $\mathfrak{n}(\mathfrak{g}_0)/\mathfrak{g}_0$, where $\mathfrak{n}(\mathfrak{g}_0)$ denotes the normalizer of $\mathfrak{g}_0$ in $\mathfrak{gl}(V)$.*

To prove Theorem 1, we will use essentially the facts that $H^1(L_{\mathfrak{sl}})\cong \boldsymbol{C}$ and $H^1(L_{\mathfrak{sp}})\cong \boldsymbol{C}$, where $L_{\mathfrak{sl}}$ (resp. $L_{\mathfrak{sp}}$) is a Lie algebra derived from $\mathfrak{sl}(n, \boldsymbol{C})$ (resp. $\mathfrak{sp}(n, \boldsymbol{C})$). (See for example [1].) The proof of Theorem 2 is carried out by elementary calculation.

It may well be doubted if every infinite transitive flat Lie algebra L has finite dimensional $H^1(L)$. But unfortunately this presumption is false. In fact we can give an easy condition for $\dim H^1(L)=\infty$. That is, we can prove

Theorem 3. *Let $L=\prod_{p\geqq -1}\mathfrak{g}_p$ be an infinite transitive flat Lie algebra which satisfies $L^{(2)}=[L^{(1)}, L^{(1)}]=0$, where $L^{(1)}=[L, L]$. Then $H^1(L)$ is "infinite" dimensional. (For such a Lie algebra L, we can construct derivations of arbitrarily large negative degree.)*

2. In this paragraph, our main objects are infinite intransitive Lie algebras $L[W^*]$. Let V and W be finite dimensional vector spaces. Put $U=V+W$ (direct sum). We denote by $S(W^*)$ the ring of formal power series over W. Let L be one of infinite transitive simple Lie algebras over V. That is, L is one of $D(V)$, $L_{\mathfrak{sl}}$, $L_{\mathfrak{sp}}$, and L_{ct} (the contact algebra). Then a Lie algebra $L[W^*]$ is obtained as a topological completion of $L\otimes S(W^*)$. These Lie algebras $L[W^*]$ are obtained as the result of the classification theorem of infinite intransitive Lie algebras, which was proved by T. Morimoto [4].

In determining $H^1(L[W^*])$, V. Guillemin's work is essential [2]. Namely we calculate the "commutator ring $\boldsymbol{C}_L$" of L and conclude that $\boldsymbol{C}_L\cong \boldsymbol{C}$. Now our result is

Theorem 4. *Let e be a trivial center of $\mathfrak{gl}(V)$. Then we have*

$$H^1(L[W^*])\cong\begin{cases} D(W) & \text{if } L=D(V) \text{ or } L_{ct}, \\ D(W)+S(W^*)e & \text{if } L=L_{\mathfrak{sl}} \text{ or } L_{\mathfrak{sp}}, \end{cases}$$

where $D(W)$ is a Lie algebra of all formal vector fields on W.

In particular, if $L=D(V)$, our result can be considered as a formal version of Y. Kanie [3].

We close our paper by giving an example of an infinite intransitive Lie algebra L with $H^1(L)=0$. Let f and g be formal functions of two variables x and y. Define a Lie algebra L by

$$L=\{f(x, y)\partial/\partial x+g(x, y)\partial/\partial y\,;\, g(x, 0)=0\}.$$

Then we have $H^1(L)=0$. Key trick depends on the fact that L contains an element $x\partial/\partial x+y\partial/\partial y$. This Lie algebra L corresponds to the group of diffeomorphisms of $\boldsymbol{R}^2$ which preserve the x-axis invariant.

References

[1] C. Freifeld: The cohomology of transitive filtered modules I: The first cohomology group. Trans. Amer. Math. Soc., **144**, 475–491 (1969).
[2] V. Guillemin: A Jordan-Hölder decomposition for a certain class of infinite

dimensional Lie algebras. J. Diff. Geometry, **2**, 313–345 (1968).
[3] Y. Kanie: Cohomologies of Lie algebras of vector fields with coefficients in adjoint representations: Foliated case. Publ. RIMS, Kyoto Univ., **14**, 487–501 (1978).
[4] T. Morimoto: On the intransitive Lie algebras whose transitive parts are infinite and primitive. J. Math. Soc. Japan, **29**, 35–65 (1977).