40. *Multiple Torsion Theories over Left and Right Perfect Rings*

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Multiple torsion theories were first discussed by Kurata [8]: An *n-fold torsion theory* is an n-tuple $(\boldsymbol{T}_1, \boldsymbol{T}_2, \cdots, \boldsymbol{T}_n)$ of classes of modules such that $(\boldsymbol{T}_i, \boldsymbol{T}_{i+1})$ is always a torsion theory. Kurata showed that there can be only four kinds of n-fold torsion theory. In this note we obtain characterizations of these various types for modules over a left and right perfect ring, the torsion theories being described in terms of properties of the partitions of the simple modules which they induce. Torsion theories over such a ring are closely related to the simple modules: Any TTF class $\boldsymbol{T}$ is both the smallest torsion class and the smallest torsion-free class containing

$$\{S \mid S \text{ is simple and } S \in \boldsymbol{T}\}.$$

The latter result is proved by the dualization of a method we used in [7] to "lift" torsion properties to a ring R from a factor ring R/I where I is right T-nilpotent.

All rings we discuss have identities and all modules are unital left modules. If $\boldsymbol{M}$ is a class of modules over a ring R, we define

$$\boldsymbol{M}^r = \{N \mid \mathrm{Hom}_R(M, N) = 0 \ \forall M \in \boldsymbol{M}\}$$
$$\boldsymbol{M}^l = \{K \mid \mathrm{Hom}_R(K, M) = 0 \ \forall M \in \boldsymbol{M}\}.$$

In most respects we adhere to the usage and conventions of [8].

Let M be a module over a ring R, I an ideal of R. We define submodules $M(\alpha)$ for all ordinals α as follows:

$$M(0) = M; \quad M(\alpha+1) = IM(\alpha); \quad M(\beta) = \bigcap_{\alpha<\beta} M(\alpha) \quad \text{if } \beta \text{ is a limit.}$$

Then for some ordinal μ we have $M(\mu+1) = M(\mu)$. If $M(\mu) = 0$, we call

$$M = M(0) \supseteq M(1) \supseteq \cdots \supseteq M(\alpha) \supseteq M(\alpha+1) \supseteq \cdots \supseteq M(\mu) = 0 \cdots (*)$$

the *descending I-series* of M.

Proposition 1. *Let M be an R-module with descending I-series* (*), *$\boldsymbol{T}$ a TTF class of R-modules. Then $M \in \boldsymbol{T}$ if and only if $M(\alpha)/M(\alpha+1) \in \boldsymbol{T}$ for each ordinal α.*

Proof. "If": If each $M(\alpha)/M(\alpha+1) \in \boldsymbol{T}$, then $M/M(1) = M(0)/M(1) \in \boldsymbol{T}$. If now $M/M(\alpha) \in \boldsymbol{T}$, it can be seen from the exact sequence

$$0 \to M(\alpha)/M(\alpha+1) \to M/M(\alpha+1) \to M/M(\alpha) \to 0$$

that $M/M(\alpha+1) \in \boldsymbol{T}$. If β is a limit and $M/M(\alpha) \in \boldsymbol{T}$ for all $\alpha < \beta$, we have an embedding

$$M/M(\beta)=M/\bigcap_{\alpha<\beta} M(\alpha)\to\prod_{\alpha<\beta} M/M(\alpha),$$

so that $M/M(\beta)\in \boldsymbol{T}$. Hence each $M/M(\alpha)\in \boldsymbol{T}$, so in particular $M\cong M/M(\mu)\in \boldsymbol{T}$.

"Only if" is clear from the closure properties of TTF classes.

Proposition 2. *The following conditions are equivalent for an ideal I of a ring R:*

(i) *Every R-module has a descending I-series.*

(ii) $IM=M\Rightarrow M=0$.

(iii) *I is left T-nilpotent.*

Proof. The equivalence of (i) and (ii) is clear, while the equivalence of (ii) and (iii) is given explicitly by Popescu and Vraciu [9], making use of ideas of Renault [10].

Although, as is well-known (see e.g. [5, Proposition 2]) a hereditary class $\boldsymbol{M}$ of R-modules determines a hereditary class $(\boldsymbol{M}^r)^l$, $(\boldsymbol{M}^l)^r$ need not be homomorphically closed, i.e. TTF, when $\boldsymbol{M}$ is. However, $(\boldsymbol{M}^l)^r$ is homomorphically closed for any non-void hereditary homomorphically closed class $\boldsymbol{M}$ of modules over a left perfect ring. We shall only be concerned with perfect rings in the sequel.

The implications (i)$\Leftrightarrow$(iii) in the following result are taken from Dickson's thesis [3].

Proposition 3. *Let $(\boldsymbol{T},\boldsymbol{F})$ be a torsion theory for modules over a left perfect ring R. The following conditions are equivalent.*

(i) *$\boldsymbol{T}$ is closed under projective covers.*

(ii) *$\boldsymbol{T}$ has a subclass $\boldsymbol{M}$ such that $\boldsymbol{T}=(\boldsymbol{M}^r)^l$ and $\boldsymbol{T}$ contains the projective covers of all modules in $\boldsymbol{M}$.*

(iii) *$\boldsymbol{F}$ is a TTF class.*

Proof. Clearly (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii): Let F be in $\boldsymbol{F}$ and let $F\to A\to 0$ be exact. Let M be in $\boldsymbol{M}$ with projective cover P. Then from the exact sequences

$$0=\mathrm{Hom}_R(P,F)\to\mathrm{Hom}_R(P,A)\to 0$$

and

$$0\to\mathrm{Hom}_R(M,A)\to\mathrm{Hom}_R(P,A)$$

we see that $\mathrm{Hom}_R(M,A)=0$. Hence $A\in \boldsymbol{M}^r=((\boldsymbol{M}^r)^l)^r=\boldsymbol{F}$.

(iii)$\Rightarrow$(i): Let $T\in\boldsymbol{T}$ have projective cover P, where $P/K\cong T$ and K is small in P. Then $P/(\boldsymbol{T}(P)+K)$, as a homomorphic image of both $P/\boldsymbol{T}(P)$ and T, belongs to $\boldsymbol{T}\cap\boldsymbol{F}=\{0\}$. Hence $P=\boldsymbol{T}(P)+K$, so $P=\boldsymbol{T}(P)\in\boldsymbol{T}$.

Corollary 4. *Let $\boldsymbol{K}$ be a non-empty, hereditary homomorphically closed class of modules over a left perfect ring R. Then $(\boldsymbol{K}^l)^r$ is a TTF class.*

Proof. Let M be in $\boldsymbol{K}^l$ and have projective cover P with $P/K\cong M$ and K small in P. If $P/L\in\boldsymbol{K}$ for some L, then $P/K+L\in\boldsymbol{K}^l\cap\boldsymbol{K}=\{0\}$

so $P=K+L=L$. Since $\boldsymbol{K}$ is hereditary, we have $P \in \boldsymbol{K}^l$. By Proposition 3, $(\boldsymbol{K}^l)^r$ is TTF.

Corollary 5. *For a left perfect ring R, the correspondence $\boldsymbol{S} \to (\boldsymbol{S}^l)^r$ defines a bijection from the sets of (non-isomorphic) simple R-modules to the TTF classes $(\neq \{0\})$ of R-modules.*

Proof. By Corollary 4, all the classes $(\boldsymbol{S}^l)^r$ are TTF, and clearly different sets $\boldsymbol{S}$ define different classes $(\boldsymbol{S}^l)^r$. On the other hand, invoking Propositions 1 and 2 when I is the Jacobson radical of R, we see that every TTF class $\boldsymbol{T} \neq \{0\}$ coincides with

$$(\{S \mid S \text{ is simple and } S \in \boldsymbol{T}\}^l)^r.$$

We now specialize further, to the case where R is left and right perfect. In this case the TTF-classes $\neq \{0\}$ are the classes $(\boldsymbol{S}^r)^l$, for sets $\boldsymbol{S}$ of simple R-modules. At least four proofs of the latter assertion have been given ([1], [6], [7], [11]). In what follows we shall find it convenient to write $\hat{\boldsymbol{S}}$ for $(\boldsymbol{S}^r)^l = (\boldsymbol{S}^l)^r$ when $\boldsymbol{S}$ is a non-empty set of simple modules. For such a set $\boldsymbol{S}$, we have two torsion theories, $(\boldsymbol{C}, \hat{\boldsymbol{S}})$ and $(\hat{\boldsymbol{S}}, \boldsymbol{D})$. Extending this usage slightly, we implicitly assign the name $\hat{\phi}$ to $\{0\}$. The classes $\boldsymbol{C}$ and $\boldsymbol{D}$ both contain all the simple modules in the complement of $\boldsymbol{S}$ (and only these). We wish, among other things, to determine when one or both of $\boldsymbol{C}$ and $\boldsymbol{D}$ is a TTF class. In what follows, sets $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ of simple modules are called *complementary* if $\boldsymbol{S}_1 \cap \boldsymbol{S}_2 = \phi$ and each simple module is isomorphic to precisely one module in $\boldsymbol{S}_1 \cup \boldsymbol{S}_2$.

Proposition 6. *Let $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ be complementary sets of simple modules over a left and right perfect ring R, $(\boldsymbol{C}, \hat{\boldsymbol{S}}_1)$ and $(\hat{\boldsymbol{S}}_1, \boldsymbol{D})$ the torsion theories associated with $\hat{\boldsymbol{S}}_1$.*

(a) *The following conditions are equivalent:*

(i) $\boldsymbol{C} = \hat{\boldsymbol{S}}_2$.

(ii) *$\boldsymbol{C}$ is hereditary.*

(iii) *$\hat{\boldsymbol{S}}_1$ contains all injective envelopes of simple modules in $\boldsymbol{S}_1$.*

(iv) *$\hat{\boldsymbol{S}}_1$ is closed under injective envelopes.*

(b) *The following conditions are equivalent:*

(i)′ $\boldsymbol{D} = \hat{\boldsymbol{S}}_2$.

(ii)′ *$\boldsymbol{D}$ is homomorphically closed.*

(iii)′ *$\hat{\boldsymbol{S}}_1$ contains all projective covers of simple modules in $\boldsymbol{S}_1$.*

(iv)′ *$\hat{\boldsymbol{S}}_1$ is closed under projective covers.*

Proof. The equivalence of (i) and (ii) follows from our remarks above. The equivalence of (ii) and (iv) is well-known (see [4]) and the proof that (iii) implies (ii) is obtained by adapting part of the proof of the latter result. Clearly (iv) implies (iii). The equivalence of (i)′, (ii)′, (iii)′ and (iv)′ is proved by analogous arguments, making use of Proposition 3 above.

Our next step is to determine when $(\hat{\boldsymbol{S}}_1, \hat{\boldsymbol{S}}_2)$ is a torsion theory where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are complementary sets of simple modules.

Lemma 7. *Let $\boldsymbol{S}_1$, $\boldsymbol{S}_2$ be complementary sets of simple modules over a left and right perfect ring R such that $\mathrm{Ext}^1_R(S_1, S_2)=0$ for each $S_1 \in \boldsymbol{S}_1$ and $S_2 \in \boldsymbol{S}_2$. Then $\mathrm{Ext}^1_R(S_1, M)=0$ whenever M is a direct sum of modules from $\boldsymbol{S}_2$ and S_1 is in $\boldsymbol{S}_1$.*

Proof. Let $M=\oplus S$, $S \in \hat{\boldsymbol{S}}_2$. Then $\Pi S/\oplus S \in \boldsymbol{S}_2$ and we have an exact sequence

$$0=\mathrm{Hom}_R(S_1, \Pi S/\oplus S)\to \mathrm{Ext}^1_R(S_1, \oplus S)$$
$$\to \mathrm{Ext}^1_R(S_1, \Pi S)\cong \Pi\, \mathrm{Ext}^1_R(S_1, S)=0.$$

Proposition 8. *Let R be a left and right perfect ring, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ complementary sets of simple R-modules such that $\mathrm{Ext}^1_R(S_1, S_2)=0$ for every $S_1 \in \boldsymbol{S}_1$, $S_2 \in \boldsymbol{S}_2$. Then $\hat{\boldsymbol{S}}_2$ contains all injective envelopes of modules in $\boldsymbol{S}_2$.*

Proof. Let $E(S_2)$ denote an injective envelope of a simple module $S_2 \in \boldsymbol{S}_2$. Let

$$0\subseteq S_2=M_1\subseteq M_2\subseteq \cdots \subseteq M_\alpha \subseteq M_{\alpha+1}\subseteq \cdots \subseteq M_\mu = E(S_2)$$

be the socle sequence of $E(S_2)$. If $M_\alpha \in \hat{\boldsymbol{S}}_2$, then for every simple submodule S of $E(S_2)/M_\alpha$ there is an exact sequence

$$0\to M_\alpha \to N \to S \to 0,$$

which splits if $S \notin \boldsymbol{S}_2$, by Lemma 7. But $S_2 \subseteq M_\alpha \subseteq N \subseteq E(S_2)$ so N is an essential extension of M_α, so S must be in $\boldsymbol{S}_2$. Hence $M_{\alpha+1}/M_\alpha = \mathrm{soc}(E(S_2)/M_\alpha) \in \hat{\boldsymbol{S}}_2$, so that $M_{\alpha+1} \in \hat{\boldsymbol{S}}_2$. If $M_\gamma \in \hat{\boldsymbol{S}}_2$ for all $\gamma<\beta$, where β is a limit, then clearly $M_\beta \in \hat{\boldsymbol{S}}_2$. It follows that $E(S_2)=M_\mu \in \hat{\boldsymbol{S}}_2$.

Theorem 9. *Let $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ be complementary sets of simple modules over a left and right perfect ring R. The following conditions are equivalent:*

(i) *$(\hat{\boldsymbol{S}}_1, \hat{\boldsymbol{S}}_2)$ is a torsion theory.*

(ii) *$\hat{\boldsymbol{S}}_1$ is closed under projective covers.*

(iii) *$\hat{\boldsymbol{S}}_2$ is closed under injective envelopes.*

(iv) *$\mathrm{Ext}^1_R(S_1, S_2)=0$ for every $S_1 \in \boldsymbol{S}_1$, $S_2 \in \boldsymbol{S}_2$.*

Proof. The equivalence of (i), (ii) and (iii) follows from Proposition 6. By Propositions 6 and 8, (iv) implies (iii). Arguing as in the proof of Lemma 3 of [2], we can show that (i) implies (iv).

We can now use properties of partitions of the set of simple modules to classify multiple torsion theories.

Theorem 10. *Let R be a left and right perfect ring. In the category of R-modules, the multiple torsion theories are characterized as follows, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ being complementary sets of simple modules, $\boldsymbol{S}$ a set of simple modules:*

(i) *4-fold of length 4: $(\boldsymbol{C}, \hat{\boldsymbol{S}}_1, \hat{\boldsymbol{S}}_2, \boldsymbol{D})$ where $\mathrm{Ext}^1_R(S_1, S_2)=0$ for each $S_1 \in \boldsymbol{S}_1$, $S_2 \in \boldsymbol{S}_2$, $\hat{\boldsymbol{S}}_1$ is not closed under injective envelopes and $\hat{\boldsymbol{S}}_2$ is not closed under projective covers.*

(ii) *3-fold of length 3, not extendable to 4-fold*: $(\boldsymbol{C}, \hat{\boldsymbol{S}}, \boldsymbol{D})$ *where* $\hat{\boldsymbol{S}}$ *is closed under neither injective envelopes nor projective covers.*

(iii) *3-fold of length 2*: $(\hat{\boldsymbol{S}}_1, \hat{\boldsymbol{S}}_2, \hat{\boldsymbol{S}}_1)$ *where both* $\hat{\boldsymbol{S}}_1$ *and* $\hat{\boldsymbol{S}}_2$ *are closed under injective envelopes and projective covers.*

Since a left and right perfect ring has the primary decomposition property if and only if all its hereditary torsion theories are centrally splitting, the following result (cf. [2]) follows from Theorem 10.

Corollary 11. *Let R be a left and right perfect ring. The following conditions are equivalent*:

(i) *R has the primary decomposition property.*

(ii) *$\hat{\boldsymbol{S}}$ is closed under injective envelopes for every set $\boldsymbol{S}$ of simple R-modules.*

(iii) *$\hat{\boldsymbol{S}}$ is closed under projective covers for every set $\boldsymbol{S}$ of simple R-modules.*

(iv) $\mathrm{Ext}^1_R(S_1, S_2)=0$ *for all non-isomorphic simple R-modules S_1 and S_2.*

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