32. *A Note on Arithmetics in Semigroups**)

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Considering a modular closure of an ideal system in a semigroup, we can define, following [1], a conductor of an order contained in (and equivalent to) a regular maximal order of the semigroup. The aim of this note is to introduce, by the conductor, regular ideals, regular v-ideals and regular subsets of the semigroup, and to give factorization theorems for these ideals and subsets by using the results in [2], [5]. Our results are applicable to the case of rings, if we take the "module-generation" as a modular closure.

Let S be a (not necessarily commutative) semigroup with unity, and let O be an order of S [2] such that it is contained in a regular maximal order E of S and equivalent to E. Then, since E is regular, O is regular and any (two-sided) E-ideal is a (two-sided) O-ideal. We now fix a closure operation: $\mathfrak{a}\mapsto\mathfrak{a}'$ of the set of all O-ideals to itself with the conditions (i) $\mathfrak{a}\subseteq\mathfrak{a}'$, (ii) $\mathfrak{a}\subseteq\mathfrak{b}$ implies $\mathfrak{a}'\subseteq\mathfrak{b}'$, (iii) $\mathfrak{a}''=\mathfrak{a}'$, and (iv) $\mathfrak{a}'\mathfrak{b}'\subseteq(\mathfrak{a}\mathfrak{b})'$. Here we assume, in addition, (v) $O'=O$, (vi) $E'=E$, and (vii) the lattice of the closed O-ideals is modular (cf. [4] for the condition (vii)). The existence of such a closure is assured by the discrete closure. Let O be an order of a ring $R=(R, +, \cdot)$ such that it is contained in (and equivalent to) a regular maximal order of R. Then for each 'semigroup O-ideal' $\mathfrak{a}$ in the semigroup $(R, \cdot)$, the map: $\mathfrak{a}\mapsto\mathfrak{a}'=$(the 'ring O-ideal' generated by $\mathfrak{a}$) satisfies the above seven conditions. By this, our results below are applicable to the case of rings.

We introduce, following [1], the *conductor* $\mathfrak{f}=\{x\in S\,;\,ExE\subseteq O\}$ of O with respect to E. Then $\mathfrak{f}$ is the unique maximal closed E-ideal (two-sided) contained in O. Now we have

(1) *If* $\mathfrak{a}$ *is an* O*-ideal with* $(\mathfrak{a}\cup\mathfrak{f})'=O$, *then* $(E\mathfrak{a}E)'$ *is an* E*-ideal satisfying* $(E\mathfrak{a}E)'=(E\mathfrak{a})'=(\mathfrak{a}E)'$, $(E\mathfrak{a}E\cup\mathfrak{f})'=E$, *and* $(E\mathfrak{a}E)'\cap O=\mathfrak{a}'$, *where* $\cup$ *and* $\cap$ *denote set-union and intersection respectively.*

(2) *If* A *is an* E*-ideal with* $(A\cup\mathfrak{f})'=E$, *then* $A\cap O$ *is an* O*-ideal satisfying* $((A\cap O)\cup\mathfrak{f})'=O$ *and* $(E(A\cap O)E)'=A'$.

For any two O-ideals $\mathfrak{a}$, $\mathfrak{b}$ we define the *join* $\mathfrak{a}\vee\mathfrak{b}$ by $(\mathfrak{a}\cup\mathfrak{b})'$, the *meet* $\mathfrak{a}\wedge\mathfrak{b}$ by the intersection of $\mathfrak{a}$ and $\mathfrak{b}$, and the *multiplication* $\mathfrak{a}\cdot\mathfrak{b}$ by $(\mathfrak{a}\mathfrak{b})'$. Then these three operations are valid for E-ideals. Henceforth, let **F** be the l.o. semigroup [6] consisting of the closed O-ideals $\mathfrak{a}$'s with

*) Dedicated to Professor G. Azumaya for his sixtieth birthday.

$\mathfrak{a}\vee\mathfrak{f}=O$, and **K** the l.o. semigroup consisting of the closed E-ideals A's with $A\vee\mathfrak{f}=E$. Then by using (1), (2) we can show that

(3) *The map* $\varphi:\mathbf{F}\to\mathbf{K};\ \mathfrak{a}\mapsto\varphi(\mathfrak{a})=E\cdot\mathfrak{a}\cdot E$ *gives an l.o. semigroup-isomorphism. Under* φ *the prime ideals in* **F** *correspond to the prime ideals in* **K**. (*The inverse of* φ *is* $A\mapsto A\wedge O$.)

A closed O-ideal $\mathfrak{a}$ is said to be *regular*, if there is an ideal $\mathfrak{c}\in\mathbf{F}$ such that $\mathfrak{c}\cdot\mathfrak{a}\in\mathbf{F}$. Then we can prove that if $\mathfrak{a}$ is regular, both $\mathfrak{b}\cdot\mathfrak{a}$ and $\mathfrak{a}\cdot\mathfrak{b}$ are members of **F** for each $\mathfrak{b}\in\mathbf{F}$ with $\mathfrak{b}\mathfrak{a}\subseteq O$. Hence in particular $\mathfrak{a}\cdot\mathfrak{c}\in\mathbf{F}$. Let **T** be the set of all regular ideals. Then we have the following:

(4) **T** *is a residuated lattice under the usual residuals*: $\mathfrak{a}/\mathfrak{b}=\{x\in S;\ x\mathfrak{b}\subseteq\mathfrak{a}\}$ *and* $\mathfrak{b}\backslash\mathfrak{a}=\{x\in S;\ \mathfrak{b}x\subseteq\mathfrak{a}\}$.

(5) *For each* $\mathfrak{a}\in\mathbf{T}$, *we have* $\mathfrak{a}/\mathfrak{a}=\mathfrak{a}\backslash\mathfrak{a}=O$.

(6) *The inverse ideal* $\mathfrak{a}^{-1}=\{x\in S;\ \mathfrak{a}x\mathfrak{a}\subseteq\mathfrak{a}\}$ *of* $\mathfrak{a}\in\mathbf{T}$ *is the set-union of the O-ideals* $\mathfrak{c}$*'s such that* $\mathfrak{a}\cdot\mathfrak{c}\cdot\mathfrak{a}\subseteq\mathfrak{a}$.

(7) *For each* $\mathfrak{a}\in\mathbf{T}$, *we have* $O/\mathfrak{a}=\mathfrak{a}\backslash O=\mathfrak{a}^{-1}\in\mathbf{T}$.

For any regular ideal $\mathfrak{a}$, $\mathfrak{a}^*$ will denote $(\mathfrak{a}^{-1})^{-1}$. Then we have that $\mathfrak{a}\in\mathbf{F}$ implies $\mathfrak{a}^*\in\mathbf{F}$. Two regular ideals $\mathfrak{a}$ and $\mathfrak{b}$ are said to be *quasi-equal* if $\mathfrak{a}^*=\mathfrak{b}^*$ (or equivalently $\mathfrak{a}^{-1}=\mathfrak{b}^{-1}$). By using (4), (5) we can see the properties mentioned from 24th line of the 13th page to 2nd line of the 14th page in [2]. Then classifying **T** by the quasi-equal relation, we have the l.o. group $\mathfrak{G}$. The coset containing $\mathfrak{a}\in\mathbf{T}$ will be denoted dy $C(\mathfrak{a})$. Here we show that the lattice $\mathfrak{G}$ is conditionally complete. Suppose that $\{C(\mathfrak{a}_\lambda);\ \lambda\in\Lambda\}$ is bounded (upper), $C(\mathfrak{a}_\lambda)\leqslant C(\mathfrak{b})$, say. Since $\mathfrak{a}_\lambda\subseteq\mathfrak{b}^*$ for all λ, there is the least upper bound $\sup_\lambda \mathfrak{a}_\lambda$, the closed O-ideal generated by the set-union of $\mathfrak{a}_\lambda$. Then, by taking $\mathfrak{c}_\lambda\in\mathbf{F}$, $\mathfrak{d}\in\mathbf{F}$ with $\mathfrak{c}_\lambda\cdot\mathfrak{a}_\lambda\in\mathbf{F}$, $\mathfrak{d}\cdot\mathfrak{b}^*\in\mathbf{F}$, we have $O=\mathfrak{d}\cdot(\mathfrak{c}_\lambda\cdot\mathfrak{a}_\lambda)\vee\mathfrak{f}\subseteq\mathfrak{d}\cdot O\cdot(\sup_\lambda\mathfrak{a}_\lambda)\vee\mathfrak{f}\subseteq\mathfrak{d}\cdot\mathfrak{b}^*\vee\mathfrak{f}=O$, $\mathfrak{d}\cdot(\sup_\lambda\mathfrak{a}_\lambda)\vee\mathfrak{f}=O$. Hence $\sup_\lambda\mathfrak{a}_\lambda$ is a member of **T**, and it is clear that $C(\sup_\lambda\mathfrak{a}_\lambda)$ is the least upper bound of $\{C(\mathfrak{a}_\lambda);\ \lambda\in\Lambda\}$. Thus by Theorem 18 in [6; Chap. V], $\mathfrak{G}$ is a commutative group. The coset $C(\mathfrak{a})$ is called *integral* if $\mathfrak{a}^*\in\mathbf{F}$. Then any two factorizations of an integral coset have the same refinement (cf. Theorem 1.1 in [2]).

A regular ideal $\mathfrak{a}$ is called here a (regular) *v-ideal* if $\mathfrak{a}^*=\mathfrak{a}$. Then we can show that any prime ideal $\mathfrak{p}\in\mathbf{F}$ is a v-ideal, if it is not quasi-equal to O. For any two regular ideals $\mathfrak{a}$, $\mathfrak{b}$ we define the *formal multiplication* of $\mathfrak{a}^*$ and $\mathfrak{b}^*$ by $\mathfrak{a}^*\circ\mathfrak{b}^*=(\mathfrak{a}^*\cdot\mathfrak{b}^*)^*=(\mathfrak{a}\cdot\mathfrak{b})^*$. Then $\{\mathfrak{a}^*;\ \mathfrak{a}\in\mathbf{T}\}$ is an l.o. group under the formal multiplication and the set-inclusion, which is isomorphic to $\mathfrak{G}$ as l.o. groups. Thus we obtain

Theorem 1 (Refinement Theorem). *For any two decompositions* $\mathfrak{a}^*=\mathfrak{a}_1^*\circ\cdots\circ\mathfrak{a}_m^*=\mathfrak{b}_1^*\circ\cdots\circ\mathfrak{b}_n^*$ *of* $\mathfrak{a}^*$ $(\mathfrak{a}\in\mathbf{F})$ *with* $\mathfrak{a}_i\in\mathbf{F}$, $\mathfrak{b}_j\in\mathbf{F}$, *there is a decomposition* $\mathfrak{a}^*=\mathfrak{c}_1^*\circ\cdots\circ\mathfrak{c}_t^*$ *such that* $\mathfrak{c}_k\in\mathbf{F}$, *and all* $\mathfrak{a}_i^*$ *and all* $\mathfrak{b}_j^*$ *appear among* $\mathfrak{c}_1^*,\cdots,\mathfrak{c}_t^*$.

Theorem 2. *If the ascending chain condition holds for v-ideals* *in* **F**, *then each v-ideal in* **F** *is factored as a "$\circ$"-product of a finite number of prime ideals in* **F**, *each of which is not quasi-equal to O. The factorization is unique within the commutativity of the formal multiplication.*

Theorem 3. *Under the same assumption as in Theorem 2, each v-ideal* $\mathfrak{a} \in \mathbf{T}$ *is decomposed as* $\mathfrak{a}=\mathfrak{p}_1^{\varepsilon_1} \circ \cdots \circ \mathfrak{p}_n^{\varepsilon_n}$, *where* $\mathfrak{p}_i$ *are different prime ideals in* **F** *such that they are not quasi-equal to O, and* ε_i *are positive or negative integers. The decomposition is unique within the commutativity of the formal multiplication.*

A subset M of S is called *regular*, if (i) for each element $x \in M$ there is a regular ideal $\mathfrak{a}$ such that $x \in \mathfrak{a}$ and $\mathfrak{a} \subseteq M$, and (ii) for any regular ideals $\mathfrak{a}$ and $\mathfrak{b}$ in M, $(\mathfrak{a} \vee \mathfrak{b})^* \subseteq M$. Suppose that the maximum condition holds for v-ideals in **F**. Then by using some results in [5], the regular sets are represented as some vectors over the set-union of the integers and $-\infty$. Hence prime spots of M are defined, and the P-component of O is defined for any subset P of prime ideals in **F**. Then we can prove the following theorem, which is a generalization of both Theorem 3 and Theorem in [3].

Theorem 4. *Under the same assumption as in Theorem 2, each regular set M in S is decomposed as*

$$M=\mathfrak{p}_1^{\alpha_1} \circ \cdots \circ \mathfrak{p}_n^{\alpha_n} \circ \left(\bigcup_{\nu} \overset{\circ}{\prod_{m(\nu)}} \mathfrak{q}_{m(\nu)}^{\beta_{m(\nu)}}\right) \circ O_P$$

where $\mathfrak{p}_i$, $\mathfrak{q}_{m(\nu)}$ *are different prime ideals in* **F** *such that they are not quasi-equal to O,* α_i *are positive integers,* $\beta_{m(\nu)}$ *are negative integers,* $\overset{\circ}{\prod}$ *denotes a finite product with respect to "$\circ$",* $\bigcup$ *denotes the set-union, P is the set of* $(-\infty)$*-prime spots of M, and* O_P *is the P-component of O. Moreover the decomposition is unique within the commutativity of the formal multiplication.*

References

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