31. *Extensions of Partially Ordered Abelian Groups*

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Let K be an abelian group and A the group of integers under addition. Let G be an abelian group extension of A by K with respect to a factor system $f: K\times K\to A$. The author proved in [4] that there exists a factor system $g: K\times K\to A$ such that $g(\alpha,\beta)\geq 0$ for all $\alpha,\beta\in K$ and g is equivalent to f. Nordahl [3] discussed the case where A is the group of real numbers. In this paper the author extends the results to the case where A is a partially ordered abelian group. The operation is additively denoted and the identity is denoted by 0. Let D be an abelian group and B a subsemigroup of D containing 0. A subsemigroup P of B is called a *cone* of B if (i) $0\in P$, and (ii) $a, -a\in P$ imply $a=0$. P induces a compatible partial order on B, and every compatible partial order on B is determined by a cone P as follows [1]:

$$x, y\in B, \quad x\geq_\rho y \quad \text{if and only if } x-y\in P.$$

The order $\geq_\rho$ is called the partial order associated with P, and (X,ρ) denotes a set X with a partial order ρ.

Let A be a subgroup of an abelian group G and let $K=G/A$, hence $G=\bigcup_{\xi\in K} A_\xi$, $A_0=A$. Let T be a subsemigroup of G containing a cone P of A such that P generates $T\cap A$. Let $T_\xi=T\cap A_\xi\neq\phi$ for each $\xi\in K$. Also assume that there is a set $\{p_\xi:\xi\in K\}$ of exactly one element p_ξ from each A_ξ such that $T_\xi=p_\xi+T_0$ for each $\xi\in K$ where $T_0=T\cap A$.

Lemma 1. *The partial order ρ_0 on T_0 associated with P can be extended to a partial order ρ on T such that $\rho=\bigcup_{\xi\in K}\rho_\xi$, $\rho_\xi=\rho|T_\xi$, and each (T_ξ,ρ_ξ) is order-isomorphic to (T_0,ρ_0).*

Proof. Since P is also a cone of T, P determines a partial order ρ on T. We see that if $a, b\in T_0$ then $a\geq_{\rho_0} b$ if and only if $p_\xi+a\geq_{\rho_\xi} p_\xi+b$. We have $\rho=\bigcup_{\xi\in K}\rho_\xi$ as desired.

Let H be an abelian group, X a cone of H and ρ the partial order on H associated with X. Then X generates H if and only if (H,ρ) is directed in the sense of Proposition 3, [1, p. 13].

A, G, K, A_ξ, A_0 are defined above. Let $g: G\to K$ be the natural homomorphism. Let P be a cone of A such that A is generated by P, and let σ_0 be the partial order on P associated with P.

Theorem 2. *There exists a partially ordered subsemigroup (S,σ) of G such that the following are satisfied:*

(2.1) $S\cap A=P$.

(2.2) *If, for each* $\xi \in g(S)$, $S_\xi = S \cap A_\xi$ *and* $\sigma_\xi = \sigma | S_\xi$, *then* (S_ξ, σ_ξ) *is order-isomorphic to* (P, σ_0) *and* $\sigma = \bigcup_{\xi \in g(S)} \sigma_\xi$.

(2.3) $g(S) = K$.

Proof. Let $\mathcal{S}$ be the set of all partially ordered subsemigroups (S, σ) of G such that (2.1)–(2.2) hold and $g(S)$ is a subgroup of K. Since $P \in \mathcal{S}$, $\mathcal{S} \neq \phi$. Define a partial order $\preccurlyeq$ on $\mathcal{S}$ by $(S_1, \sigma_1) \preccurlyeq (S_2, \sigma_2)$ if and only if (i) $g(S_1) \subseteq g(S_2)$, (ii) $\alpha \in g(S_1)$ implies $S_1 \cap A_\alpha = S_2 \cap A_\alpha$, (iii) $\sigma_2 | S_1 = \sigma_1$. Since $\mathcal{S}$ satisfies Zorn's property, there exists a maximal element $(\bar{S}, \bar{\sigma})$ in $\mathcal{S}$. Let $\bar{S}_\xi = \bar{S} \cap A_\xi$, $\bar{S}_0 = \bar{S} \cap A = P$, $\bar{\sigma} = \bigcup_{\xi \in g(\bar{S})} \bar{\sigma}_\xi$ where $\bar{\sigma}_\xi = \bar{\sigma} | \bar{S}_\xi$. Suppose $g(\bar{S}) \neq K$. Let $H = g(\bar{S})$, $\alpha \in K \backslash H$.

Case I. In case $i \cdot \alpha \notin H$ for all non-zero integers i. Pick $p \in A_\alpha$ and $q \in A_{-\alpha}$. Then $p + q \in A$. Since P generates A, there is an $a \in P$ such that $p + q + a \in P$. Let $r = p + a$, $s = q + a$. Obviously $r \in A_\alpha$ and $s \in A_{-\alpha}$ but $i \cdot r \notin \bar{S}$ and $i \cdot s \notin \bar{S}$ for all integers $i \neq 0$. Let T be the subsemigroup of G generated by $\bar{S}$, r and s. Let $\langle \alpha \rangle$ be the infinite cyclic subgroup of K generated by α. As $H \cap \langle \alpha \rangle = \{0\}$, we have $g(T) = H \oplus \langle \alpha \rangle$, thus $g(T)$ is a subgroup of K. Every $x \in T$ has the form: Either $x = y + i \cdot r$ or $x = y + i \cdot s$ where $y \in \bar{S}$ and i is a nonnegative integer. Both i and $\xi = g(y)$ are uniquely determined by x. In particular if $x \in T \cap A$, then $i = 0$, so $x \in \bar{S} \cap A$, hence $T \cap A = \bar{S} \cap A = P$. Since 0 is the $\bar{\sigma}_0$-least element of $\bar{S}_0 = P$, $\bar{S}_\xi$ has the $\bar{\sigma}_\xi$-least element $\bar{p}_\xi$ for each $\xi \in g(\bar{S})$. Every element y of $\bar{S}$ has a unique form $y = \bar{p}_\xi + z$ for $\xi \in g(\bar{S})$, $z \in P$. Let $T_\eta = T \cap A_\eta$ for each $\eta \in g(T)$. Let $x \in T$. If $x = y + i \cdot r$, $T_{\xi + i \cdot \alpha} = T \cap A_{\xi + i \cdot \alpha} = \bar{p}_\xi + i \cdot r + P$. If $x = y + i \cdot s$, $T_{\xi - i \cdot \alpha} = T \cap A_{\xi - i \cdot \alpha} = \bar{p}_\xi + i \cdot s + P$. By Lemma 1 we have an extension τ of $\bar{\sigma}$ to T such that $\tau = \bigcup_{\eta \in g(T)} \tau_\eta$, $\tau_\eta = \tau | T_\eta$ and (T_η, τ_η) is order-isomorphic to (P, σ_0) for each $\eta \in g(T)$.

Case II. In case $i_0 \cdot \alpha \in H$ for some positive integer $i_0 > 1$. (If $i_0 < 0$, take $-\alpha$ instead of α_1.) Assume i_0 is the smallest of such, i.e., $i \cdot \alpha \notin H$ if $i < i_0$ but $i_0 \cdot \alpha \in H$. Let $p \in A_\alpha$. Then $i_0 \cdot p \in A_{i_0 \cdot \alpha}$. If $\bar{p}_{i_0 \cdot \alpha}$ denotes the $\bar{\sigma}_{i_0 \cdot \alpha}$-least element of $\bar{S}_{i_0 \cdot \alpha}$, then $i_0 \cdot p - \bar{p}_{i_0 \cdot \alpha} \in A$, so there is an $a \in P$ such that $i_0 \cdot p - \bar{p}_{i_0 \cdot \alpha} + a \in P$, hence $i_0 \cdot p + a \in \bar{S}_{i_0 \cdot \alpha}$. Let $q = p + a$. Clearly $i \cdot q \notin \bar{S}$ for all $i < i_0$ but $i_0 \cdot q \in \bar{S}$. Let T be the subsemigroup of G generated by $\bar{S}$ and q. Every element x of T has the form $x = y + i \cdot q$, $0 \leq i < i_0$ where $y \in \bar{S}$. It is easy to see that $g(T)$ is a subgroup of K and $T \cap A = P$. Since $\bar{S}_\xi = \bar{p}_\xi + P$, $T \cap A_{\xi + i \cdot \alpha} = (\bar{p}_\xi + i \cdot q) + P$ for each $\xi \in g(\bar{S})$ where $\bar{p}_\xi$ is the $\bar{\sigma}_\xi$-least element of $\bar{S}_\xi$. By Lemma 1 there is an extension τ of $\bar{\sigma}$ to T satisfying the same conditions as in Case I.

In both Cases I and II, T satisfies (2.1), (2.2) and $g(T)$ is a subgroup of K, hence $T \in \mathcal{S}$. But $\bar{S} \subsetneq T$. This contradicts the maximality of $\bar{S}$. Therefore $H = K$. This completes the proof.

Theorem 3. *Let K be an abelian group, and P a cone of A. Moreover, assume P generates A. If G is an abelian group extension* *of A by K with respect to a factor system $f: K\times K\to A$, then there is a factor system $g: K\times K\to A$ such that*

(3.1) *$g(\xi,\eta)\in P$ for all $\xi,\eta\in K$.*

(3.2) *g is equivalent to f.*

Proof. Let $G=\{(x,\xi): x\in A, \xi\in K\}$ where $(x,\xi)+(y,\eta)=(x+y+f(\xi,\eta),\xi+\eta)$. As $f(0,0)=0$, A is identified with $\{(x,0): x\in A\}$ under $x\to(x,0)$. By Theorem 2, there is a subsemigroup S of G satisfying (2.1)–(2.3). Let $S=\bigcup_{\xi\in K}S_\xi$, $S_0=P$. Recall σ_0 is the partial order on P associated with P, and σ is the extension of σ_0 to S and $\sigma_\xi=\sigma|S_\xi$, $\sigma=\bigcup_{\xi\in K}S_\xi$ namely $(x,\xi)\geq_\sigma(y,\eta)$ if and only if $\xi=\eta$ and $x-y\in P$. Let (p_ξ,ξ) be the σ_ξ-least element of S_ξ. If $\xi=0$, $p_0=0$ since $S_0=P$. For $(p_\xi,\xi)\in S_\xi$, $(p_\eta,\eta)\in S_\eta$, we have

$$(p_\xi,\xi)+(p_\eta,\eta)=(p_\xi+p_\eta+f(\xi,\eta),\xi+\eta)\in S_{\xi+\eta}.$$

Since $(p_{\xi+\eta},\xi+\eta)$ is the $\sigma_{\xi+\eta}$-least element of $S_{\xi+\eta}$,

$$(p_\xi+p_\eta+f(\xi,\eta),\xi+\eta)\geq_{\sigma_{\xi+\eta}}(p_{\xi+\eta},\xi+\eta)$$

whence $p_\xi+p_\eta+f(\xi,\eta)-p_{\xi+\eta}\in P$. Note p_ξ is a function $K\to A$. Let $g(\xi,\eta)=p_\xi+p_\eta-p_{\xi+\eta}+f(\xi,\eta)$. Then $g(\xi,\eta)\in P$ for all $\xi,\eta\in K$ and $g(0,0)=0$ since $p_0=0$. Thus g is a factor system $K\times K\to P$ and it is equivalent to f.

Remark. In the proof of Theorem 5 in [4], the author defined g by $g(\alpha,\beta)=f(\alpha,\beta)+\delta(\alpha)+\delta(\beta)-\delta(\alpha\beta)$. In order to make g a factor system, we should define δ' by $\delta'(\alpha)=0$ if $\alpha=\varepsilon$; $\delta'(\alpha)=\delta(\alpha)$ if $\alpha\neq\varepsilon$; and define $g'(\alpha,\beta)=f(\alpha,\beta)+\delta'(\alpha)+\delta'(\beta)-\delta'(\alpha\beta)$. The author is grateful to Dr. Nordahl.

References

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