29. *A Modification of Cayley's Family of Cubic Surfaces and Birational Action of $W(E_6)$ over It*

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Cayley introduced in [2] the following family of cubic surfaces in $P_3:(x:y:z:w)$ with parameters l, m, n, k

$$(1)\quad w\Big[x^2+y^2+z^2+w^2+\Big(mn+\frac{1}{mn}\Big)yz+\Big(ln+\frac{1}{ln}\Big)xz+\Big(lm+\frac{1}{lm}\Big)xy+\Big(l+\frac{1}{l}\Big)xw+\Big(m+\frac{1}{m}\Big)yw+\Big(n+\frac{1}{n}\Big)zw\Big]+kxyz=0$$

in order to show that there are exactly 45 tritangent planes[1] for the general cubic surface. He proved this by introducing new parameter q by $k=\frac{1}{q}(q+lmn)\Big(q+\frac{1}{lmn}\Big)$ and by writing all the 45 tritangents in the form of linear homogeneous equations in x, y, z, w of which the coefficients depend rationally on the parameters (l, m, n, q). In modern language this means exactly that, if the family is regarded as a fiber space over the base (l, m, n, q), the monodromy group associated with the second (integral) cohomology of the fiber is trivial. This fact further suggests that we could modify the family (1), within a quotient by finite morphism, so that the Weyl group $W(E_6)$ of the simple Lie group E_6 may act, at least birationally, over the modification. (The evidence consists in the following well known fact: The homology classes of the 27 lines upon the cubic surface generate the cohomology group via Poincaré duality, and further the intersection form, when restricted to the orthogonal complement of the class of hyperplane-section, is just isomorphic to the Killing form of E_6 restricted to the module generated by roots.) The purpose of this note is to show that this can actually be done.

We first introduce a new system $(X:Y:Z:W)$ of homogeneous coordinates of P_3 by setting

$$x=l^2mnX,\quad y=lm^2nY,\quad z=lmn^2Z,\quad w=-lmnkW/q.$$

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1) A tritangent plane, or a *tritangent* for short, means a projective plane which meets the surface in the union of three lines. Since exactly 5 tritangents pass through each of the lines upon the surface, it follows that famous theorem: There are exactly 27 lines upon the general cubic surface.

Then the equation of the original family (1) is changed to the following

$$(2)\quad \rho W[\lambda X^2+\mu Y^2+\nu Z^2+(\rho-1)^2(\lambda\mu\nu\rho-1)^2W^2 \\ +(\mu\nu+1)YZ+(\lambda\nu+1)XZ+(\lambda\mu+1)XY \\ -(\rho-1)(\lambda\mu\nu\rho-1)W\{(\lambda+1)X+(\mu+1)Y+(\nu+1)Z\}]+XYZ=0$$

where we have set

$$\lambda=l^2,\quad \mu=m^2,\quad \nu=n^2,\quad \rho=-1/lmnq.$$

We can thus consider the new family (2) of cubic surfaces to be defined over the new base space (λ,μ,ν,ρ). (Precisely speaking, the fiber space (2) is just the quotient of the fiber space over the base (l,m,n,q) by the action of the finite group $\{\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3):\varepsilon_i=\pm 1\ (i=1,2,3)\}$ defined by

$$\varepsilon(l,m,n,q\,;x:y:z:w) \\ =(\varepsilon_1 l,\varepsilon_2 m,\varepsilon_3 n,\varepsilon_1\varepsilon_2\varepsilon_3 q\,;\varepsilon_2\varepsilon_3 x:\varepsilon_1\varepsilon_3 y:\varepsilon_1\varepsilon_2 z:\varepsilon_1\varepsilon_2\varepsilon_3 w).)$$

We are now able to define a birational action of $W(E_6)$ over the modified family (2). Recall that $W(E_6)$ is defined, as a Coxeter group, by generators $s_1, s_2, \cdots, s_6$ with the commutation relation expressed by the following diagram:

$$\begin{array}{ccccccccc} s_1 & — & s_3 & — & s_4 & — & s_5 & — & s_6 \\ & & & & | & & & & \\ & & & & s_2 & & & & \end{array}$$

(see Bourbaki [1, p. 20–22]). It suffices therefore to construct birational transformations $\varphi_1,\varphi_2,\cdots,\varphi_6$ of the total space $(\lambda,\mu,\nu,\rho,X,Y,Z,W)$ which send λ,μ,ν,ρ to rational functions of them, which are linear with respect to X,Y,Z,W, which make the equation (2) invariant and which satisfy the required commutation relation so that the correspondence $s_i\mapsto\varphi_i$ $(i=1,2,\cdots,6)$ defines the action of $W(E_6)$ over (2). We set

$$\varphi_2:\ \begin{cases} \lambda\to\lambda & X\to\mu\{X+\rho(\mu-1)(\nu-1)W\} \\ \mu\to\mu^{-1} & Y\to\mu^2 Y \\ \nu\to\nu & Z\to\mu\{Z+\rho(\lambda-1)(\mu-1)W\} \\ \rho\to\mu\rho & W\to\mu W \end{cases}$$

$$\varphi_3:\ \begin{cases} \lambda\to\lambda^{-1} & X\to\lambda^2 X \\ \mu\to\mu & Y\to\lambda\{Y+\rho(\lambda-1)(\nu-1)W\} \\ \nu\to\nu & Z\to\lambda\{Z+\rho(\lambda-1)(\mu-1)W\} \\ \rho\to\lambda\rho & W\to\lambda W \end{cases}$$

$$\varphi_5:\ \begin{cases} \lambda\to\lambda & X\to\nu\{X+\rho(\mu-1)(\nu-1)W\} \\ \mu\to\mu & Y\to\nu\{Y+\rho(\lambda-1)(\nu-1)W\} \\ \nu\to\nu^{-1} & Z\to\nu^2 Z \\ \rho\to\nu\rho & W\to\nu W \end{cases}$$

$$\varphi_4:\ \begin{cases} \lambda\to\lambda\rho & X\to\rho^{-2}\{X-(\rho-1)(\mu\nu\rho-1)W\} \\ \mu\to\mu\rho & Y\to\rho^{-2}\{Y-(\rho-1)(\lambda\nu\rho-1)W\} \\ \nu\to\nu\rho & Z\to\rho^{-2}\{Z-(\rho-1)(\lambda\mu\rho-1)W\} \\ \rho\to\rho^{-1} & W\to\rho^{-1}W \end{cases}$$

$$\varphi_1:\quad \begin{cases} \lambda\to\dfrac{\lambda\mu\nu\rho^2(1-\lambda)}{\lambda\mu\nu\rho^2-1} \\ \mu\to\dfrac{(\lambda\mu\rho-1)(\lambda\mu\nu\rho-1)}{\mu(\lambda\rho-1)(\lambda\nu\rho-1)} \\ \nu\to\dfrac{(\lambda\nu\rho-1)(\lambda\mu\nu\rho-1)}{\nu(\lambda\rho-1)(\lambda\mu\rho-1)} \\ \rho\to\dfrac{(\lambda\rho-1)(\lambda\mu\nu\rho^2-1)}{\rho(\lambda-1)(\lambda\mu\nu\rho-1)} \\ X\to-\dfrac{(\lambda\mu\nu\rho^2-1)(\lambda^2\mu\nu\rho^2-1)}{\rho^2(\lambda-1)(\lambda\mu\nu\rho-1)^2}X \\ Y\to\dfrac{\mu\nu(\lambda\rho-1)(\lambda^2\mu\nu\rho^2-1)}{\rho(\lambda-1)(\lambda\mu\rho-1)(\lambda\mu\nu\rho-1)^2}\{\lambda X+\lambda\mu\rho Y+Z-(\rho-1)(\lambda\mu\nu\rho-1)W\} \\ Z\to\dfrac{\mu\nu(\lambda\rho-1)(\lambda^2\mu\nu\rho^2-1)}{\rho(\lambda-1)(\lambda\nu\rho-1)(\lambda\mu\nu\rho-1)^2}\{\lambda X+Y+\lambda\nu\rho Z-(\rho-1)(\lambda\mu\nu\rho-1)W\} \\ W\to\dfrac{\mu\nu(\lambda\rho-1)}{(\lambda\mu\nu\rho-1)(\lambda^2\mu\nu\rho^2-1)}\{\lambda X-(\lambda\rho-1)(\lambda\mu\nu\rho-1)W\} \end{cases}$$

$\varphi_6=\tau\circ\varphi_1\circ\tau$

where τ denotes the obvious automorphism of the family (2) interchanging λ,ν and X, Z and leaving the remaining variables unchanged. For the shortness of this note, it has to be left to the reader to check that the transformations thus defined actually have the desired properties. Presumably we should explain why the transformations φ_2, $\varphi_3, \varphi_4, \varphi_5$ can be divided by obvious factors $\mu, \lambda, \rho^{-1}, \nu$ which are not necessary from the projective point of view. These factors were just required in order that $\varphi_1, \varphi_2, \cdots, \varphi_6$ might, even from the linear point of view, satisfy the commutation relation.

We finally remark that λ, μ, ν, ρ can be considered to form a fundamental system of roots of the simple group D_4 when regarded as charactors on the torus $T=\{(\lambda, \mu, \nu, \rho)\,;\, \lambda\mu\nu\rho\neq 0\}$, which is then to be regarded as the maximal torus of the group D_4 of the adjoint type. The subgroup generated by $\varphi_2, \varphi_3, \varphi_4, \varphi_5$, acting biregularly over T, is nothing but the Weyl group $W(D_4)$. (Observe the part of the diagram above constituted by s_2, s_3, s_4, s_5.)

For the problem how to refine and extend this birational action of $W(E_6)$ up to the biregular action over some reasonable moduli space of cubic surfaces we refer the reader to the forthcoming paper of the first author [3].

References

[1] Bourbaki, N.: Groupes et Algèbres de Lie. Chaps. 4, 5, 6, Herman, Paris (1968).
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[3] Naruki, I.: Cross ratios as moduli of cubic surfaces. Proc. Japan Acad., **56A**, 126–129 (1980).