4. On the Initial Boundary Value Problem of the Linearized Boltzmann Equation in an Exterior Domain

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1. Problem and result. Let O be a bounded convex domain in $\boldsymbol{R}^n$ $(n\geq 3)$ with a smooth boundary and $\Omega=\boldsymbol{R}^n\backslash\bar{O}$. Put $Q=\Omega\times\boldsymbol{R}^n$ and $S^{\pm}=\{(x,\xi)\in\partial\Omega\times\boldsymbol{R}^n\,;\,n(x)\cdot\xi\gtrless 0\}$, where $n(x)$ is the inner normal of $\partial\Omega$ at x. For $u=u(t,x,\xi)$ which is related to the density of gas particles at time $t\geq 0$ and a point $x\in\Omega$ with a velocity $\xi\in\boldsymbol{R}^n$, our equation is described as follows;

(1.1) $\dfrac{\partial u}{\partial t}=-\sum_{j=1}^{n}\xi_j\dfrac{\partial u}{\partial x_j}-\nu(\xi)u+\int_{\boldsymbol{R}^n}K(\xi,\eta)u(t,x,\eta)d\eta.$

(1.2) $u|_{s^+}=C(u|_{s^-}).$

(1.3) $u|_{t=0}=u_0(x,\xi).$

Here C is a linear operator from a function space on S^- to the similar one on S^+. Our assumptions on the collision operator $L=\nu(\xi)-K$ are those of cut-off hard potentials.

(1.4) *$\nu(\xi)$ is continuous in ξ, depends only on $|\xi|$ and $\nu(\xi)\geq\nu_0>0$ for some constant ν_0.*

(1.5) *$K(\xi,\eta)=K(\eta,\xi)$ is real valued and continuous for $\xi\neq\eta$, $\int_{\boldsymbol{R}^n}|K(\xi,\eta)|^p\,d\eta<\infty$ for some p, $1<p<\infty$, $\int_{\boldsymbol{R}^n}|K(\xi,\eta)|\,(1+|\eta|)^{-\alpha}d\eta\leq d_\alpha(1+|\xi|)^{-\alpha-1}$ for any $\alpha\geq 0$.*

Moreover the operator L is non-negative self-adjoint in $L^2(\boldsymbol{R}^n)$ and has an isolated eigenvalue 0 with eigenfunctions $\{1,\xi_1,\cdots,\xi_n,|\xi|^2\}\times\exp\left(-\dfrac{1}{2}|\xi|^2\right)$. (Note that the operator K induced from the integral kernel $K(\xi,\eta)$ is a compact self-adjoint operator in $L^2(\boldsymbol{R}^n)$.)

As for the operator C we assume

(1.6) $\|C\|\leq 1$

as an operator from $L^2(S^-\,;\rho)$ to $L^2(S^+\,;\rho)$, where $\rho=\rho(x,\xi)=|n(x)\cdot\xi|$ and $L^2(S^{\pm}\,;\rho)$ is the space of square integrable function on $S^{\pm}$ with respect to the measure $\rho(x,\xi)dS_x d\xi$.

We define the linearized Boltzmann operator B by

(1.7) $B=-\sum_{j=1}^{n}\xi_j\dfrac{\partial}{\partial x_j}-\nu(\xi)+K=-\xi\cdot\nabla_x-L$ with domain $D(B)$

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$=\{u \in L^2(Q)\,;\,(\xi\cdot\nabla_x+\nu(\xi))u(x,\xi)\in L^2(Q)$ and $u|_{s+}=Cu|_{s-}\}$. Then we have the following

Theorem. *Let $n\geq 3$. Assume the conditions* (1.4)–(1.6). *Then, the linearized Boltzmann operator B generates a contraction semi-group e^{tB}. Moreover, if $u_0\in L^2(Q)\cap L^2(\boldsymbol{R}^n\,;\,L^1(\Omega))$, we have*

(1.8) $\|e^{tB}u_0\|_{L^2(Q)}\leq c\ \ a_n(t)\ \ (\|u_0\|_{L^2(Q)}+\|u_0\|_{L^2(\boldsymbol{R}^n;\,L^1(\Omega))})$, *where* $a_3(t)=(1+t)^{-3/4}\log(2+t)$, $a_4(t)=(1+t)^{-1}\log(2+t)$ *and* $a_n(t)=(1+t)^{-1}$ *for* $n\geq 5$.

Remark. The exterior initial-boundary value problem of the linearized Boltzmann equation was first considered by Ukai [6] in case of reverse reflection. He obtained similar estimates for $n\geq 5$ with $a_n(t)=(1+t)^{-1/2}$. The estimate (1.8) enables us to study the global existence of the solution of the exterior initial-boundary value problem of the non-linear Boltzmann equation [1].

2. Results on the case $Q^\infty=\boldsymbol{R}^n\times\boldsymbol{R}^n$ without the boundary condition. Put $A^\infty=-\xi\cdot\nabla_x-\nu(\xi)$ and $B^\infty=A^\infty+K$, with $D(A^\infty)=D(B^\infty)=\{u\in L^2(Q^\infty)\,;\,(\xi\cdot\nabla_x+\nu(\xi))u(x,\xi)\in L^2(Q^\infty)\}=D^\infty$. A^∞ generates a semi-group in $X=L^2(Q^\infty)$, $(e^{tA^\infty}u)(x,\xi)=e^{-t\nu(\xi)}u(x-t\xi,\xi)$ and $\|e^{tA^\infty}\|\leq e^{-t\nu_0}$. B^∞ also generates a contraction semi-group in X. Put $\hat{u}(k,\xi)=\int e^{-ik\cdot x}u(x,\xi)dx$, $A^\infty(k)=i\xi\cdot k-\nu(\xi)$ and $B^\infty(k)=A^\infty(k)+K$. Then $(A^\infty u)^\wedge(k,\xi)=A^\infty(k)\hat{u}(k,\xi)$ and $(B^\infty u)^\wedge(k,\xi)=B^\infty(k)\hat{u}(k,\xi)$. $(\lambda-A^\infty(k))^{-1}K$ is a compact operator in $L^2(\boldsymbol{R}^n)$ and depends continuously on λ and k. If $\mathrm{Re}\,\lambda>-\nu_0$ and $\{1-(\lambda-A^\infty(k))^{-1}K\}^{-1}$ exists (is uniformly bounded for $k\in\boldsymbol{R}^n$), then $(\lambda-B^\infty(k))^{-1}=\{1-(\lambda-A^\infty(k))^{-1}K\}^{-1}(\lambda-A(k))^{-1}$ exists ($(\lambda-B^\infty)^{-1}$ exists).

Ukai [5] and Nishida-Imai [4] proved that $\|(\sigma+i\tau-A^\infty(k))^{-1}K\|\to 0$ as $|\tau|+|k|\to\infty$ uniformly in $\sigma\geq-\nu_1$, $0<\nu_1<\nu_0$.

Thus, putting $C(-\nu_1,\tau_1)=\{\sigma+i\tau\,;\,\sigma\geq 0,\ \tau\in\boldsymbol{R}\}\cup\{\sigma+i\tau\,;\,-\nu_1\leq\sigma\leq 0,\ |\tau|\geq\tau_1\}$ for some $\nu_1<\nu_0$ and $\tau_1>0$, we have that $\|\{1-(\lambda-A^\infty(k))^{-1}K\}^{-1}\|\leq C_0'$ for $(\lambda,k)\in C(-\nu_1,\tau_1)\times\boldsymbol{R}^n$. $B^\infty(k)$ is maximal dissipative in $L^2(\boldsymbol{R}^n)$ and has no eigenvalues on the imaginary axis for $k\neq 0$. This fact and following Lemma 2.1 imply that the resolvent set $\rho(B^\infty)$ of B^∞ contains $C(-\beta_\infty,\tau_\infty)\cup\{\sigma+i\tau\,;\,-a_\infty\tau^2\leq\sigma\leq 0,\ |\tau|\leq\tau_\infty\}\backslash\{0\}\equiv\sum(\beta_\infty,a_\infty)\backslash\{0\}$, for some $\nu_0>\beta_\infty>0$, $a_\infty>0$ with $\beta_\infty=a_\infty\tau_\infty^2$.

Lemma. 2.1 (Ellis-Pinsky [2]). *There exists $\kappa_0>0$ such that if $|k|\leq\kappa_0$ and* $\mathrm{Re}\,\lambda>0$,

(2.1) $$(\lambda-B^\infty(k))^{-1}=\sum_{j=0}^{n+1}\frac{1}{\lambda-\lambda_j(k)}P_j(k)+(\lambda-B^\infty(k))^{-1}P(k).$$

$\lambda_j(k)$'s are C^∞ functions of k and

(2.2) $\lambda_j(k)=\pm i|k|\lambda_j^{(1)}-|k|^2\lambda_j^{(2)}+O(|k|^3)$

with $\lambda_j^{(1)}$ real and $\lambda_j^{(2)}>0$. $P_j(k)$'s are also C^∞ functions of k and one-dimensional projections commuting with $B^\infty(k)$, $P(k)=1-\sum P_j(k)$.

$(1+|\xi|)^{\alpha}P_j(k)$ and $P_j(k)(1+|\xi|)^{\alpha}$ are bounded operators in $L^2(\boldsymbol{R}^n)$. $(\lambda-B^{\infty}(k))^{-1}P(k)$ is analytically continued to $\{\mathrm{Re}\,\lambda\geq-\sigma_1\}$ with $\sigma_1>0$ and uniformly bounded in $k\in R^n$ and λ there.

Put $P'(k)=\sum P_j(k)$, $P'u=(2\pi)^{-n}\int_{|k|\leq k_0} e^{ik\cdot x}\hat{u}(k,\xi)dk$ and $P=1-P'$. Denote by $\|\ \|$ the norm in X. Let $X_1=L^2(\boldsymbol{R}^n_\xi;L^1(\boldsymbol{R}^n_x))$ with norm $\|\ \|_1$, and $X_\infty=L^2(\boldsymbol{R}^n_\xi;|\xi|d\xi;L^\infty(\boldsymbol{R}^n_x))$ with norm $\|\ \|_\infty$.

Lemma 2.2 (cf. [4] and [5]). *For $u\in X\cap X_1$,*

(2.2) $\|e^{tB^{\infty}}P'u\|\leq C_0(1+t)^{-n/4}\|u\|_1$,

(2.3) $\|e^{tB^{\infty}}Pu\|\leq C_0e^{-\sigma_1 t}\|u\|$,

(2.4) $\|e^{tB^{\infty}}P'u\|_\infty\leq C_0(1+t)^{-n/4}\|u\|$,

(2.5) $\|(\lambda-B^{\infty})^{-1}P'u\|_\infty\leq C_0(1+|\lambda|)^{-1}\|u\|_1$, $\lambda\in\sum(\beta_\infty,a_\infty)$,

(2.6) $\|(\lambda-B^{\infty})^{-2}P'u\|_\infty<C_0b_n(|\lambda|)\|u\|_1$, $\lambda\in\sum(\beta_\infty,a_\infty)$,

where $b_3(s)=s^{-1/2}$, $b_4(s)=\log(1+s^{-1})$, and $b_n(s)=1$, $n\geq 5$.

(2.7) $\int_{-\infty}^{\infty}\|(\sigma+i\tau-B^{\infty})^{-1}Pu\|^2\,d\tau\leq C_0\|u\|^2$, $\sigma\geq-\beta_\infty$.

All these estimates hold for $B^{\infty *}=\xi\cdot\nabla_x-\nu(\xi)+K$.

3. Exterior problem. Let X and X_1 be as in § 2 with $\boldsymbol{R}^n_x$ replaced by Ω. Let $D=\{u\in X;(\xi\cdot\nabla_x+\nu(\xi))u\in X\}$ and $Y_\pm=L^2(S^\pm;\rho)$ with norm $|\ |_\pm$. Any $u\in D$ has its trace $\gamma^\pm u$ on $S^\pm$, that is,

$$|\gamma^-u|_-\leq 2\|u\|\,\|(\xi\cdot\nabla_x+\nu(\xi))u\|,$$
$$|\chi(\xi)\gamma^+u|_+\leq C_\chi\|u\|\,\|(\xi\cdot\nabla_x+\nu(\xi))u\|,$$

where $\chi(\xi)$ is a bounded function with compact support.

Define a closed linear operator $A=-\xi\cdot\nabla_x-\nu(\xi)$ with $D(A)=\{u\in D;\gamma^+u=C\gamma^-u\}$. $\|C\|\leq 1$ implies that for $u\in D(A)$

(3.1) $\mathrm{Re}\,(Au,u)\leq-\nu_0\|u\|^2$.

Next two lemmas are useful to show that A generates a semi-group in $L^2(Q)$.

Lemma 3.1. *Let X and X_1 be as in § 2.* (i) *For $\sigma>-\nu_0$ and $u\in X$, there hold*

(3.2) $|\gamma^\pm(\sigma+i\tau-A^{\infty})^{-1}u|^2_\pm\leq\dfrac{2}{\sigma+\nu_0}\|u\|^2$,

(3.3) $\int_{-\infty}^{\infty}|\gamma^\pm(\sigma+i\tau-A^{\infty})^{-1}u|^2_\pm\,d\tau=2\pi\int_0^{\infty}e^{-2\sigma t}|\gamma^\pm e^{tA^{\infty}}u|^2_\pm\,dt\leq 2\pi\|u\|^2$.

(ii) *For $\lambda\in\sum(\beta_\infty,a_\infty)$ and $u\in X\cap X_1$, there hold*

(3.4) $|\gamma^\pm(\lambda-B^{\infty})^{-1}P'u|_\pm\leq C_1(1+|\lambda|)^{-1}\|u\|_1$,

(3.5) $|\gamma^\pm(\lambda-B^{\infty})^{-2}P'u|_\pm\leq C_1b_n(|\lambda|)\|u\|_1$.

(iii) For $\sigma>-\beta_\infty$ and $u\in X$, there hold

(3.6) $|\gamma^\pm(\sigma+i\tau-B^{\infty})^{-1}Pu|_\pm\leq C_1\|u\|$,

(3.7) $\int_{-\infty}^{\infty}|\gamma^\pm(\sigma+i\tau-B^{\infty})^{-1}Pu|^2_\pm\,d\tau\leq C_1\|u\|^2$.

Let $S^+(\xi)=\{x\in\partial\Omega;n(x)\cdot\xi>0\}$ and $\Omega^+(\xi)=\{x+t\xi;x\in S^+(\xi)$ and

$t>0\}$. For $g(x,\xi)\in Y_+$ we define $R(\lambda)g\in X$ by

$$(R(\lambda)g)(x+t\xi,\xi)=\begin{cases}e^{-\lambda t}e^{-\nu(\xi)t}g(x,\xi), & x+t\xi\in\Omega^+(\xi),\\ 0, & \text{elsewhere}.\end{cases}$$

For $\sigma>-\nu_0$, $R(\sigma+i\tau)$ is a bounded operator from Y_+ to X with norm $\leq(2(\sigma+\nu_0))^{-1/2}$.

Lemma 3.2. *Let $\lambda=\sigma+i\tau$, $\sigma>-\nu_0$. Then we have* (i) $R(\lambda)g\in D$, (ii) $(\lambda+\xi\cdot\nabla_x+\nu(\xi))R(\lambda)g=0$ *in* Q, (iii) $\gamma^+R(\lambda)g=g$ *and* $\gamma^-R(\lambda)g=0$, (iv) $\||\xi|^{-1/2}R(\lambda)g_1\|\leq|\partial\Omega|^{1/2}(\sigma+\nu_0)^{-1}|g|_+$, (v) $\|KR(\lambda)g\|_1\leq d_1|\partial\Omega|^{1/2}(\sigma+\nu_0)^{-1}|g|_+$ *and* (vi) $R(\lambda)^*=\gamma^+(\bar{\lambda}-A^{\infty*})^{-1}$. *Here $|\partial\Omega|$ is the measure of $\partial\Omega$.*

For a function u on Q, let eu be the extension of u to Q^∞, by putting $eu=0$ outside Q. Denote by rv the restriction of a function v on Q^∞ to Q. We can easily see that

(3.8) $(\lambda-A)^{-1}=r(\lambda-A^\infty)^{-1}e-R(\lambda)(\gamma^+-C\gamma^-)(\lambda-A^\infty)^{-1}e$ for $\operatorname{Re}\lambda>-\nu_0$. Because of (3.1), A generates a semi-group in X and $\|e^{tA}\|\leq e^{-t\nu_0}$.

Since K is a bounded operator in X, $B=A+K$ generates a semi-group in X. The inequality $\operatorname{Re}(Bu,u)\leq 0$ implies $\|e^{tB}\|\leq 1$. By the resolvent equation we have

$$(\lambda-B)^{-1}=\{1-(\lambda-A)^{-1}K\}^{-1}(\lambda-A)^{-1}.$$

From the next lemma we see that for any $\nu_2\in(0,\nu_0)$ there exists $\tau_2>0$ such that $\|\{1-(\lambda-A)^{-1}K\}^{-1}\|\leq C_2$ for $\lambda\in C(-\nu_2,\tau_2)$.

Lemma 3.3. *$\gamma^\pm(\lambda-A^\infty)^{-1}K$ is a compact operator from $L^2(Q^\infty)$ to $Y_\pm$. For any $\nu_2\in(0,\nu_0)$, $\|(\sigma+i\tau-A^\infty)^{-1}K\|\to 0$ as $|\tau|\to\infty$ uniformly in $\sigma\geq-\nu_2$.*

Now we note another expression of $(\lambda-B)^{-1}$;

(3.9) $(\lambda-B)^{-1}=r(\lambda-B^\infty)^{-1}e-R(\lambda)(\gamma^+-C\gamma^-)(\lambda-B^\infty)^{-1}e-(\lambda-B)^{-1}\times KR(\lambda)(\gamma^+-C\gamma^-)(\lambda-B^\infty)^{-1}e$.

Putting $\tilde{C}=\gamma^+-C\gamma^-$, we have

(3.10) $\{1+(\lambda-B)^{-1}K\}\{1+R(\lambda)\tilde{C}(\lambda-B^\infty)^{-1}Ke\}=1+r(\lambda-B^\infty)^{-1}Ke$.

Put $p=er$, $p'=1-p$ and $B_O^\infty=A^\infty+Kp'=B^\infty-Kp$. Then we have $\{1-(\lambda-A^\infty)^{-1}Kp'\}\{1+(\lambda-B_O^\infty)^{-1}Kp'\}=1$ and $\{1-(\lambda-B_O^\infty)^{-1}Kp\}\{1+(\lambda-B^\infty)^{-1}Kp\}=1$ in $L^2(Q^\infty)$. $(\lambda-A^\infty)^{-1}Kp'$ is a compact operator in $L^2(Q^\infty)$ and B_O^∞ has no eigenvalues on the imaginary axis. Therefore from the similar argument of § 1, $(\lambda-B_O^\infty)^{-1}$ exists for $\operatorname{Re}\lambda\geq-\nu_3$, $0<\nu_3<\nu_0$, as an operator in $L^2(Q^\infty)$. Thus $\{1+(\lambda-B^\infty)^{-1}Kp\}^{-1}=\{1-(\lambda-B_O^\infty)^{-1}Kp\}$ exists for $\operatorname{Re}\lambda\geq-\nu_3$. Hence $\{1+r(\lambda-B^\infty)^{-1}Ke\}^{-1}$ exists for $\operatorname{Re}\lambda\geq-\nu_3$ as an operator in $X=L^2(Q)$. Thus $\{1-(\lambda-A)^{-1}K\}^{-1}=\{1+(\lambda-B)^{-1}K\}$ exists if and only if $\{1+R(\lambda)\tilde{C}(\lambda-B^\infty)^{-1}K\}^{-1}$ exists, when $\operatorname{Re}\lambda\geq-\nu_3$. (3.9), (3.10) and the operator equality $(1+TU)^{-1}=1-T(1+UT)^{-1}U$ imply that

(3.11) $(\lambda-B)^{-1}=r(\lambda-B^\infty)^{-1}e-[\gamma^+(\lambda-B^{\infty*})^{-1}e]^*\{1+\tilde{C}(\lambda-B^\infty)^{-1}\times KeR(\lambda)\}^{-1}\tilde{C}(\lambda-B^\infty)^{-1}e$.

Putting $T(\lambda)=\tilde{C}(\lambda-B^{\infty})^{-1}KeR(\lambda)$, we have

Lemma 3.4. (i) *$T(\lambda)$ is a compact operator on Y_+ for $\lambda\in\sum(\beta_\infty, a_\infty)$.* (ii) *$T(\lambda)$ and $T'(\lambda)$ are analytic in $\sum(\beta_\infty, a_\infty)\backslash\{0\}$ and*

$$(3.12)\quad \|T(\lambda)\|\leq C_3,\ \left\|\frac{d}{d\lambda}T(\lambda)\right\|\leq C_3 b_n(|\lambda|).$$

(iii) *There exists $\sum(\beta, a)\subset\sum(\beta_\infty, a_\infty)$ such that for $\lambda\in\sum(\beta, a)$ $\{1+T(\lambda)\}^{-1}$ exists and*

$$(3.13)\quad \|\{1+T(\lambda)\}^{-1}\|\leq C_3.$$

Let $U(t)$ be the inverse Laplace transform of $\{1+T(\lambda)\}^{-1}\tilde{C}(\lambda-B^{\infty})^{-1}e$. Then (3.11) implies that for u and $v\in X$

$$(e^{tB}u, v)=(re^{tB^{\infty}}eu, v)-\int_0^t\langle U(s)u, \gamma^+e^{(t-s)B^{\infty*}}ev\rangle_+ds.$$

From (2.4) and (3.7) for $B^{\infty*}$, it follows that

$$|\gamma^+e^{tB^{\infty*}}P'^*ev|_+\leq C_0|\partial\Omega|^{1/2}(1+t)^{-n/4}\|v\|,$$

$$\int_0^\infty e^{2\sigma_0 t}|\gamma^+e^{tB^{\infty*}}P^*ev|_+^2\,dt\leq C_1\|v\|^2.$$

Put $U(t)=U_1(t)+U_2(t)$, where

$$U_1(t)=\frac{1}{2\pi i}\int_{\Gamma_1}e^{\lambda t}\{1+T(\lambda)\}^{-1}\tilde{C}(\lambda-B^{\infty})^{-1}ed\lambda,$$

$$U_2(t)=\frac{1}{2\pi i}\int_{\Gamma_2}e^{\lambda t}\{1+T(\lambda)\}^{-1}\tilde{C}(\lambda-B^{\infty})^{-1}ed\lambda,$$

with $\Gamma_1=\{\lambda=-a\tau^2+i\tau\,;\ |\tau|\leq\tau_0\}$ and $\Gamma_2=\{\lambda=-\beta+i\tau\,;\ |\tau|\geq\tau_0,\ \beta=a\tau_0^2\}$. Then on account of Lemma 3.4, we get

$$|U_1(t)u|_+\leq C_4(1+t)^{-1}\|u\|_1,$$

$$\int_0^\infty e^{2\beta t}|U_2(t)|_+^2\,dt\leq C_4\|u\|^2.$$

Thus we have

$$|(e^{tB}u, v)|\leq C_0(1+t)^{-n/4}(\|u\|_1+\|u\|)\|v\|+C_5a_n(t)(\|u\|_1+\|u\|)\|v\|,$$

which proves (1.8).

Remark. Recently we obtained decay estimates for solutions of Cauchy problem of the linearized Boltzmann equation with a cut-off soft potential.

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