28. *On Curvatures of Homogeneous Convex Cones*

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1. It is known that homogeneous convex cones play an important role in the theory of homogeneous bounded domains (see e.g., [3], [5], [8], [10], [12]). From the differential geometric point of view, it is interesting to investigate the Riemannian geometric properties of homogeneous convex cones. Several results about homogeneous self-dual cones are known. For instance, a homogeneous self-dual cone is a Riemannian symmetric space of non-positive curvature [9]. However it is little known about homogeneous non-self-dual cones. In this note, we will announce some results about the Riemannian geometry of homogeneous convex cones. The detailed results with their complete proofs will appear elsewhere.

2. Let V be an open convex cone in the n-dimensional real number space $\boldsymbol{R}^n$ which does not contain any full straight line. We denote by $G(V)$ the group of all linear automorphisms of V, that is, $G(V) = \{a \in GL(n)\,;\, aV = V\}$. If $G(V)$ acts transitively on V, then the cone V is called *homogeneous*. Let $\langle\ ,\ \rangle$ be an inner product in $\boldsymbol{R}^n$. Then the *dual cone* V^* of V is defined by $V^* = \{y \in \boldsymbol{R}^n\,;\, \langle x, y\rangle > 0$ for any x in $\bar{V} - (0)\}$, where $\bar{V}$ is the topological closure of V in $\boldsymbol{R}^n$. A cone V is called *self-dual* if the dual cone V^* with respect to a suitable inner product coincides with V. Following Koecher and Vinberg, we define the *characteristic function* φ_V of V by

$$\varphi_V(x) = \int_{V^*} \exp - \langle x, y\rangle dy \qquad (x \in V),$$

where dy is a canonical Euclidean measure on $\boldsymbol{R}^n$. From the characteristic function of V, we define a symmetric 2-form g on V by

$$g = \sum_{i,j} \frac{\partial^2 \log \varphi_V}{\partial x_i \partial x_j} dx_i dx_j,$$

where $(x_1, x_2, \cdots, x_n)$ denotes a linear coordinates of $\boldsymbol{R}^n$. Then g is a $G(V)$-invariant Riemannian metric on V, which is called a *canonical Riemannian metric* of V. Therefore with this metric, the cone V is a homogeneous Riemannian manifold (cf. [9], [10], [12]).

3. In this section, we state results about the canonical Riemannian metric. It was proved in [11] that for every positive constant c, the surface in $\boldsymbol{R}^n$ defined by $\{x \in V\,;\, \varphi_V(x) = c\}$ is a homogeneous affine hypersphere of hyperbolic type. By using this, we can prove the fol-

lowing

Proposition 1. *A homogeneous convex cone is homothetically equivalent to a product Riemannian manifold of a homogeneous affine hypersphere of hyperbolic type and a* 1*-dimensional flat space.*

It is known in [2] that the Ricci curvature of a complete affine hypersphere of hyperbolic type is non-positive. Combining this and Proposition 1, we have

Theorem 1. *The Ricci curvature of a homogeneous convex cone is non-positive.*

We remark that for the sectional curvature, the analogous assertion as in the theorem mentioned above does not hold. In fact if $n \geq 8$, then there exists an n-dimensional homogeneous convex cone whose sectional curvature attains both signs. For instance, we have

Proposition 2. *Let V be a homogeneous convex cone in $\boldsymbol{R}^{7+m}$ defined as follows*: $V=\{x=(x_1, x_2, \cdots, x_{7+m})\,;\, x_4>0, A(x)$ *is positive definite*$\}$, *where $A(x)=(a_{ij}(x))$ is a symmetric matrix of degree* 3 *such that* $a_{11}(x) = x_1x_4 - \sum_{8 \leq i \leq 7+m} x_i^2$, $a_{12}(x)=x_4x_5$, $a_{13}(x)=x_4x_6$, $a_{22}(x)=x_2x_4$, $a_{23}(x)=x_4x_7$ *and* $a_{33}(x)=x_3x_4$. *Then the sectional curvature of V attains both signs.*

On the other hand, lower dimensional homogeneous convex cones were classified in [6]. The following theorems are proved by using this classification and calculations based on the methods in [1], [4] and [7].

Theorem 2. *Let V be an n-dimensional homogeneous convex cone with $n \leq 7$. Then the sectional curvature of V is non-positive.*

On isometries of the canonical Riemannian metric, we have

Theorem 3. *Let V be an n-dimensional homogeneous convex cone with $n \leq 8$. Then there exists no infinitesimal isometry other than infinitesimal linear isometry on V.*

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