26. *Lévy's Functional Analysis in Terms of an Infinite Dimensional Brownian Motion. I*

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§ 1. **Introduction.** In his book [1], Paul Lévy has extensively developed a potential theory in an infinite dimensional space.

T. Hida and H. Nomoto have constructed the projective limit $(\mathring{S}_\infty, \mu)$ of the topological stochastic family $\{(\mathring{S}_n, \mu_n)\}$ consisting of the open subsets $\mathring{S}_n$ of the finite dimensional spheres S_n and the restrictions μ_n to $\mathring{S}_n$ of the uniform probability measures on S_n such that $\mu_n(\mathring{S}_n)=1$.

By using this theory, we shall prove the relation:

$$L^2(\mathring{S}_\infty, \mu)=\varprojlim L^2(\mathring{S}_n, \mu_n),$$

and give an interpretation to Lévy's potential theory for Dirichlet problems on the unit ball by introducing the Brownian motion $(\boldsymbol{B}, \boldsymbol{E})$ on an infinite dimensional space $\boldsymbol{E}$ such that $E\supset\mathring{S}_\infty$. We shall also establish the strong Markov property, the uniform continuity of the paths and the skew product formula of the Brownian motion.

§ 2. **Projectively consistent construction of multiple Wiener integrals.** First we reformulate T. Hida and H. Nomoto's results [2] in a slightly different manner from theirs. Let S_n be the sphere with center zero and radius $\sqrt{n+1}$ in the $(n+1)$-dimensional Euclidean space E_{n+1}, and $\mathring{S}_n$ be the open subset of S_n consisting of the points $(x_1, \cdots, x_{n+1})$:

$$\begin{cases} x_1=\sqrt{n+1}\prod_{i=1}^{n}\sin\theta_i, \\ x_k=\sqrt{n+1}\cos\theta_{k-1}\prod_{i=k}^{n}\sin\theta_i, \qquad (k=2, \cdots, n), \\ x_{n+1}=\sqrt{n+1}\cos\theta_n, \end{cases}$$

with the restriction that $(\theta_1, \cdots, \theta_n)\in\Pi^n$, where $\Pi^n=\{(\theta_1, \cdots, \theta_n);\ 0<\theta_1<2\pi,\ 0<\theta_i<\pi,\ i=2, \cdots, n\}$. We denote by $\pi_{n,m}$ $(n>m)$ the projection of $\mathring{S}_n$ to $\mathring{S}_m$ such that the following is commutative:

$$\begin{array}{ccc} \Pi^n \ni (\theta_1, \cdots, \theta_n) & \longrightarrow & (\theta_1, \cdots, \theta_m)\in \Pi^m \\ \downarrow & & \downarrow \\ \mathring{S}_n & \xrightarrow{\quad \pi_{n,m} \quad} & \mathring{S}_m. \end{array}$$

Set

$$\mathring{S}_\infty=\left\{x=(x_1, \cdots, x_n, \cdots);\ \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}x_k^2=1,\ x_1\neq 0 \text{ or } x_2<0\right\}$$

and define a sequence of projections $\{p_n; n\geqslant 1\}$ by

$$p_n x = x_n \qquad \text{for } x=(x_1, \cdots, x_n, \cdots)\in \mathring{S}_\infty.$$

We denote by $\mathring{\mathcal{S}}_\infty$ the σ-algebra generated by cylinder sets of $\mathring{S}_\infty$ and by μ the standard Gaussian white noise on $\mathring{S}_\infty$. Then $\{p_n; n\geqslant 1\}$ on $(\mathring{S}_\infty, \mu)$ can be viewed as mutually independent random variables which are subject to the standard normal distribution $N(0,1)$.

Further we define the projection $\pi_n; \mathring{S}_\infty \to \mathring{S}_n$ as follows:

$$\pi_n x = \frac{1}{\|x\|_{n+1}}(x_1, \cdots, x_{n+1}),$$

where $\|x\|_{n+1}^2 = \frac{1}{n+1}\sum_{k=1}^{n+1} x_k^2$. Then we have the following

Proposition 2.1 (T. Hida and H. Nomoto [2]). *Let $\mathring{\mathcal{S}}_n$ $(n\geqslant 1)$ be the topological σ-algebra on $\mathring{S}_n$, and μ_n be the restriction to $\mathring{S}_n$ of the uniform probability measure on S_n. Then*

1) $\bigcup_{n=1}^{\infty} \pi_n^{-1}(\mathring{\mathcal{S}}_n)$ *generates the σ-algebra $\mathring{\mathcal{S}}_\infty$,*
2) $\mu_n(A)=\mu(\pi_n^{-1}(A))$ *for* $A\in\mathring{\mathcal{S}}_n$,
3) $\mu_n(\pi_{n,m}^{-1}(A))=\mu_m(A)$, $(n>m)$ *for* $A\in\mathring{\mathcal{S}}_m$.

Let $\mathcal{H}_n$ $(1\leqslant n<\infty)$ be the sequence of the complex Hilbert spaces $L^2(\mathring{S}_n, \mathring{\mathcal{S}}_n, \mu_n)$. We shall define a projection $\rho_{n,m}$ $(n>m)$ of $\mathcal{H}_n$ to $\mathcal{H}_m$ using the branching rule of the representation theory of the rotation group $SO(n+1)$ (see [5, pp. 449–451]).

To begin with, for integers $j\geqslant 2$, $m\geqslant k\geqslant 0$ we put

$$D_{j,k,m}(\theta)=A_{j,k,m}C_{m-k}^{(j-1)/2+k}(\cos\theta)(\sin\theta)^k,$$

where $C_{m-k}^{(j-1)/2+k}$ denotes the Gegenbauer polynomial and the positive constant $A_{j,k,m}$ is determined so as to have

$$\int_0^\pi D_{j,k,m}^2(\theta)(\sin\theta)^{j-1}d\theta=\int_0^\pi(\sin\theta)^{j-1}d\theta.$$

A base of homogeneous harmonic polynomials on $\mathring{S}_n$ can be taken to be the family

$$\Xi_{K_n,\pm}^n(\theta_1, \cdots, \theta_n)=e^{\pm ik_1\theta_1}\prod_{j=2}^n D_{j,k_{j-1},k_j}(\theta_j),$$

where K_n stands for the sequence of integers $(k_1, \cdots, k_n)$,

$$0\leqslant k_1\leqslant k_2<\cdots\leqslant k_n.$$

Since the system $\bigcup_{K_n}\{\Xi_{K_n,+}^n, \Xi_{K_n,-}^n\}$ constitutes a C.O.N.S. in $\mathcal{H}_n$, we can determine the orthogonal projection $\rho_{n,m}$ $(n>m)$ of $\mathcal{H}_n$ to $\mathcal{H}_m$ in terms of $\Xi_{K_n,\pm}^n$'s:

$$\rho_{n,m}\Xi_{K_n,\pm}^n=\begin{cases}\Xi_{K_m,\pm}^m & \text{if } k_m=k_{m+1}=\cdots=k_n\\ 0 & \text{otherwise,}\end{cases}$$

where K_m denotes the subsequence $(k_1, \cdots, k_m)$ of the given sequence K_n. Thus the projective system $\{\mathcal{H}_n, \rho_{n,m}\}$ has been defined.

Now for an infinite sequence K of integers $(k_1, \cdots, k_n, \cdots)$ satisfying $0\leqslant k_1\leqslant\cdots\leqslant k_{n-1}\leqslant k_n=k_{n+1}=k_{n+2}=\cdots$, with an n, we define the functions $\Xi_{K,+}, \Xi_{K,-}$ on $\mathring{S}_\infty$ as follows:

$$\Xi_{K,\pm}(x)=e^{\pm ik_1\theta_1}\prod_{j=2}^{n} D_{j,k_{j-1},k_j}(\theta_j)\left(\frac{\Gamma((n+1)/2)}{\Gamma(|K|+(n+1)/2)}\left(\frac{n+1}{2}\right)^{|K|}\right)^{1/2}\|x\|_{n+1}^{|K|}.$$

where $|K|$ and $(\theta_1, \cdots, \theta_n)$ denote k_n and the Euler angles of $\pi_n x$ for the n, respectively. Since the family $\bigcup_K \{\Xi_{K,+}, \Xi_{K,-}\}$ constitutes a C.O.N.S. in the complex Hilbert space $\mathcal{H}=L^2(\mathring{S}_\infty, \mu)$, we can define the orthogonal projection ρ_n of $\mathcal{H}$ to $\mathcal{H}_n$ as follows:

$$\rho_n \Xi_{K,\pm}=\begin{cases}\Xi^n_{K_n,\pm} & \text{if } k_n=k_{n+1}=k_{n+2}=\cdots \\ 0 & \text{otherwise,}\end{cases}$$

where K_n denotes the subsequence $(k_1, \cdots, k_n)$ of the given infinite sequence K. Now, dualizing Proposition 2.1, we have the following

Theorem 2.2. 1) *For* $f \in \mathcal{H}$,

$$\lim_{n\to\infty}\int_{S_\infty} |f(x)-(\rho_n f)(\pi_n x)|^2\,\mu(dx)=0.$$

2) *Let* $\{f_n \in \mathcal{H}_n; n\geqslant 1\}$ *be a projectively consistent sequence, that is,* $\rho_{n,m} f_n=f_m$ $(n>m)$. *Then there exists a function* $f \in \mathcal{H}$ *such that*

$$\rho_n f=f_n \qquad (n\geqslant 1),$$

if and only if $\sup_n \|f_n\|_n<\infty$, *where* $\|\cdot\|_n$ *denotes the norm of* $\mathcal{H}_n$.

§ 3. Infinite dimensional space E and Brownian motion on E.

Set

$$E=\left\{x=(x_1, \cdots, x_n, \cdots)\in R^\infty;\ \sup_n \frac{1}{n}\sum_{k=1}^{n} x_k^2<\infty\right\},$$

and introduce semi-norms $\{\|\cdot\|_n;\ 1\leqslant n\leqslant\infty\}$:

$$\|x\|_n^2=\frac{1}{n}\sum_{k=1}^{n} x_k^2, \qquad \|x\|_\infty=\overline{\lim_{n\to\infty}}\,\|x\|_n.$$

Let O_1 and O_2 be the topologies induced by the semi-norms $\{\|\cdot\|_n;\ 1\leqslant n\leqslant\infty\}$ and the semi-norms $\{\|\cdot\|_n;\ 1\leqslant n<\infty\}$ respectively. The σ-algebra generated by cylinder sets of E will be denoted by $\mathcal{E}$: $\mathcal{E}=\sigma(p_n;\ n\geqslant 1)$, where $p_n x=x_n$ for $x=(x_1, \cdots, x_n, \cdots)\in E$.

Remark. 1) $\mathcal{E}$ is generated by O_2-open sets, not by O_1-open sets.

2) The topological space (E, O_1) is non-separable.

3) For any positive number α and x in E,

$$\sum_{n=1}^{\infty}\frac{x_n^2}{n(\log n)^{1+\alpha}}<\infty.$$

In the sequel we shall use only the O_1-topology of E without explicit mentions. Now let $\{w_n(t);\ n\geqslant 1\}$ be a family of mutually independent 1-dimensional Wiener processes $(w_n(0)=0)$ on a complete probability space $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$. Then we can prove

Theorem 3.1. 1) *For any* $x=(x_1, \cdots, x_n, \cdots)\in E$,

$$\overline{\lim_{n\to\infty}}\,\frac{1}{n}\sum_{k=1}^{n}(x_k+w_k(t))^2=\|x\|_\infty^2+t, \qquad \textit{for any } t\geqslant 0 \textit{ a.s.}$$

2) $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(w_k(t)-w_k(s))^2=|t-s|$ *for any* $t, s\geqslant 0$ *a.s.*

Thus we see that the process $W(t)$

$$W(t)=(w_1(t),\cdots,w_n(t),\cdots)$$

is living in $\boldsymbol{E}$, which will be called the ***E*-valued *Wiener process.*** We denote by Ω the totality of continuous sample paths ω on $\boldsymbol{E}$ and by B_t the mapping:

$$B_t(\omega)=\omega(t) \qquad \text{for } \omega\in\Omega.$$

Now Theorem 3.1 gives us probability measures P^x on Ω, $x\in\boldsymbol{E}$, such that:

$$P^x(B(t_k)\in A_k,\ k=1,\cdots,n)=\hat{P}(W(t_k)\in A_k-x,\ k=1,\cdots,n)$$

for $0\leqslant t_1<\cdots<t_n<\infty$ and $A_1,\cdots,A_n\in\mathcal{E}$. Then we have a strong Markov process (Ω, P^x, B_t) with state space $(\boldsymbol{E},\mathcal{E})$. Theorem 3.1 shows

(3.1) $\|B(t,\omega)\|_\infty^2=\|B(0,\omega)\|_\infty^2+t,$ for any $t\geqslant 0$ a.s. P^x, $x\in\boldsymbol{E}$.

Hence denoting by τ the first exit time from the unit ball $D_\infty=\{x\in\boldsymbol{E};\ \|x\|_\infty<1\}$, we have

$$\tau(\omega)=1-\|x\|_\infty^2 \qquad \text{a.s. } P^x,\ (x\in D_\infty).$$

We shall call the process $\boldsymbol{B}=(\Omega,P^x,B_t)$ with state space $(\boldsymbol{E},\mathcal{E})$ *the Brownian motion on* $\boldsymbol{E}$.

§4. Spherical Brownian motion. In this section we shall see that as in finite dimensional cases, the Brownian motion $\boldsymbol{B}$ on $\boldsymbol{E}$ is factored as the skew product of its radial part and an independent spherical Brownian motion (see [3]). We denote by S_∞ the unit sphere $\{x\in\boldsymbol{E};\ \|x\|_\infty=1\}$, and $\tilde{P}^\xi$, $(\xi\in S_\infty)$ the restriction of the probability measure P^ξ to the set $\tilde{\Omega}=\{\omega\in\Omega;\ \|\omega(0)\|_\infty=1\}$. In view of (3.1), by putting

$$\xi_t(\omega)=e^{-t/2}B(e^t-1,\omega),$$

we have a strong Markov process $(\tilde{\Omega},\tilde{P}^\xi,\xi_t)$ with state space S_∞. We also have for any $\xi\in S_\infty$

$$\|\xi(t,\omega)-\xi(s,\omega)\|_\infty^2=2(1-e^{-|t-s|/2}) \qquad \text{for any } t,s\geqslant 0 \text{ a.s. } \tilde{P}^\xi.$$

We shall call this process the *spherical Brownian motion.* Let $\dot{P}^r$ $(r\in(0,\infty))$ be the probability measure on the set $\dot{\Omega}=(0,\infty)$ such that

$$\dot{P}^r(A)=\begin{cases}1 & \text{if } r\in A\\ 0 & \text{if } r\notin A.\end{cases}$$

We define the mapping r_t as follows: $r_t(\dot{\omega})=\sqrt{\dot{\omega}^2+t}$ for $\dot{\omega}\in\dot{\Omega}$. Then we have a deterministic Markov process $(\dot{\Omega},\dot{P}^r,r_t)$ with state space $(0,\infty)$, which will be called the *radial process.* Associate the *random clock* τ_t:

$$\tau_t(\dot{\omega})=\int_0^t\frac{ds}{r_s^2(\dot{\omega})}=\log\frac{\dot{\omega}^2+t}{\dot{\omega}^2}$$

with the radial process. By putting $\hat{\boldsymbol{E}}=\{x\in\boldsymbol{E};\ \|x\|_\infty>0\}$, we have

Theorem 4.1. *The skew product process* $(\hat{\Omega},\hat{P}^x,\hat{B}_t)$ *with state space* $\hat{\boldsymbol{E}}$:

$$\hat{\hat{B}}_t((\dot{\omega}, \omega)) = r_t(\dot{\omega})\xi(\tau_t(\dot{\omega}), \omega),$$
$$P^x = P^r \times P^\xi \qquad \text{for } x = r\xi,\ r>0,\ \xi \in S_\infty,\ x \in \boldsymbol{E},$$

is equivalent to the part of the Brownian motion $\boldsymbol{B}$ *on the open set* $\hat{\boldsymbol{E}}$.

It seems that this theorem could be identified with the Lévy's formula ([1, p. 305, (5)]) which is expressed in terms of the generators of the Brownian motion, the radial process and the spherical Brownian motion.

Lastly we remark that the spherical Brownian motion $\xi(t)$ on S_∞ is an infinite dimensional Ornstein-Uhlenbeck process. The theorem gives a new approach to investigation of infinite dimensional Brownian motions and Ornstein-Uhlenbeck processes (cf. [4]). More details will be discussed in a forthcoming note.

References

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