# 25. *Monodromy Preserving Deformation and its Application to Soliton Theory*

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§1. **Introduction.** In a preceding article [6], the author investigated the monodromy preserving deformation theory of linear differential equations. The purpose of the present note is to study its relation with the theory of isospectral deformation. In this connection, the reader is referred to the works of Ablowitz *et al.* [1], [2], and to the recent paper of Flaschka-Newell [3] in which they study the link between monodromy and spectrum preserving deformations by a slightly different approach from the present work. Here we show that the soliton theory is naturally incorporated within the framework of the former by considering a degenerate case rather than the "generic" case discussed in [6].

To be specific, the equations dealt with in this paper are the following $2\times 2$ first order systems

(1.1) $PY=0, \qquad P=d/dx-(x^{-2}E+x^{-1}F+G+\sum_{j=1}^{N} H_j/(x-a_j))$

(1.2) $PY=0, \qquad P=d/dx-(xG+F+\sum_{j=1}^{N} H_j/(x-a_j))$

where the eigenvalues of $H_j$ are now assumed to differ by integers. The deformation equations for (1.1)-(1.2) are constructed in a parallel way as in [6]. We give a necessary and sufficient condition for (1.1), (1.2) to be deformed without altering the Stokes multipliers, the global monodromy and the connection matrices, and state that the resulting non-linear equations are completely integrable (Theorems 1, 2), In §4, we sketch the proof of Theorem 1. In §5, we show that the $N$-soliton solutions for the sine-Gordon equation are related to the solution of the deformation equations for (1.1).

Further results along the present line will be published elsewhere.

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§2. **Construction of the deformation equations for (1.1).** Let $U$ be an open set in $C^p$. The $2\times 2$ coefficient matrices $E$, $F$, $G$, and $H_j$ $(1\leqq j\leqq N)$ of (1.1) are assumed to be holomorphic in $t=(t_1, \cdots, t_p)$ $\in U$. Note that (1.1) has irregular singularities of rank one at $x=0$, $\infty$, and regular ones at $x=a_j$ $(1\leqq j\leqq N)$. We make the following assumptions.

(I) $G=\operatorname{diag}(g_1(t), g_2(t))$, $E=K\tilde{E}K^{-1}$ with a diagonal matrix $\tilde{E}=\operatorname{diag}(e_1(t), e_2(t))$, and $K$ is holomorphic in $t$.

(II) $g_1(t)$, $g_2(t)$ and $e_1(t)$, $e_2(t)$ are mutually distinct, respectively.

(III) $H_j=T_j \operatorname{diag}(0,1)T_j^{-1}$ with some $T_j$ holomorphic in $U$, $1\leqq j\leqq N$.

(IV) $a_j$ $(1\leqq j\leqq N)$ are mutually distinct non-zero constants.

Here $\operatorname{diag}(\alpha,\beta)$ denotes a diagonal matrix whose entries are $\alpha$ and $\beta$. Notice that the exponents at the regular singular points $a_j$ are not "generic".

Let $\tilde{Y}(x,t)=\hat{Y}(x,t)x^{D^{(\infty)}(t)}\exp(xG(t))$ be the normalized formal matrix solution at $x=\infty$ of (1.1), and $Y_l(x,t)$ $(1\leqq l\leqq 3)$ the normalized matrix solutions at $x=\infty$ of (1.1) in the sense of [6]. To consider the asymptotic expansion at $x=0$, we make a transformation $Y=KZ$. Then (1.1) is converted into

(2.1) $dz/dx=(x^{-2}\tilde{E}+x^{-1}K^{-1}FK+K^{-1}GK+\sum_{j=1}^{N}K^{-1}H_jK/(x-a_j))Z.$

For this equation, we have the normalized formal matrix solution at $x=0$, $\tilde{Z}(x,t)=\hat{Z}(x,t)x^{D^{(0)}(t)}\exp(-x^{-1}\tilde{E}(t))$, and the normalized matrix solutions at $x=0$, $Z_l(x,t)$ $(1\leqq l\leqq 3)$. We define the Stokes multipliers $C_l^{(\infty)}$, $C_l^{(0)}$ $(1\leqq l\leqq 2)$ by

(2.2) $$Y_{l+1}=Y_lC_l^{(\infty)},\qquad Z_{l+1}=Z_lC_l^{(0)}.$$

Near $x=a_j$, equation (1.1) has a fundamental solution matrix of the form

(2.3) $$Y_{a_j}(x,t)=T_j(x-a_j)^J\Phi_j(x,t)(x-a_j)^{L_j(t)}$$
$$J=\operatorname{diag}(0,1),\qquad L_j(t)=\begin{bmatrix}0 & 0\\ l_j(t) & 0\end{bmatrix}$$

where $\Phi_j(x,t)$ is a $2\times 2$ holomorphic matrix near $x=a_j$ such that $\Phi_j(a_j,t)=I$. We define the connection matrices $Q_j$ $(0\leqq j\leqq N)$ by

(2.4) $$Y_1=KZ_1Q_0,\quad Y_1=Y_{a_j}Q_j,\quad 1\leqq j\leqq N.$$

We set the "deformation properties" as follows:

(DP.I) $$dD^{(\infty)}=dD^{(0)}=0,\quad dC_l^{(\infty)}=dC_l^{(0)}=0,\quad 1\leqq l\leqq 2.$$

(DP.II) $$dQ_0=0.$$

(DP.III) $$d(Q_j^{-1}L_jQ_j)=0,\quad (I-J)dQ_j\cdot Q_j^{-1}J=0,\quad 1\leqq j\leqq N.$$

Here $d$ denotes the exterior differentiation with respect to the parameter $t$. Note that the regular singularities $a_j$ are kept fixed.

We state

**Theorem 1.** *The deformation properties* (DP.I)–(DP.III) *hold if and only if* $G$, $F$, $\tilde{E}$, $\tilde{K}$ *and* $H_j$ $(1\leqq j\leqq N)$ *satisfy the following non-linear system*

(2.5) $$dP=[\Omega,P],\qquad d\Omega=\Omega\wedge\Omega,$$

(2.6) $$dK=K\{d\tilde{E},K^{-1}FK\}_{\tilde{E}}+\{dG,F+\textstyle\sum_{j=1}^{N}H_j\}_GK,$$

*where* $\Omega=x\Phi+\Psi+x^{-1}\Theta$ *is a* $2\times 2$ *matrix of* 1-*forms given by*

(2.7) $$\Phi=dG,\quad \Psi=\{dG,F+\textstyle\sum_{j=1}^{N}H_j\}_G,\quad \Theta=-Kd\tilde{E}K^{-1}.$$

*And the above system is equivalently rewritten into the following completely integrable system*

$$\begin{aligned}(2.8)\qquad & dK=K\{d\tilde{E},K^{-1}FK\}_{\tilde{E}}+\{dG,F+\textstyle\sum_{j=1}^{N}H_j\}_G K,\\ & dF=[\Phi,E]+[\Theta,G]+[\Psi,F]-\textstyle\sum_{j=1}^{N}a_j^{-1}[\Theta,H_j],\\ & dH_j=[\Omega|_{x=a_j},H_j],\qquad 1\leqq j\leqq N.\end{aligned}$$

Here the bracket notation $\{\ \}$ and $dP$ were introduced in our previous note [6]. We note that (2.6) is obtained as the integrability condition for the extended system $(K^{-1}PK)Z_1=0$, $dZ_1=(K^{-1}\Omega K-K^{-1}dK)Z_1$, and that $G$, $\tilde{E}$ can be regarded as independent variables.

**§3. Construction of the deformation equations for (1.2).** The deformation theory for (1.2) can be established in a parallel way as in §2. In what follows, we give an outline of the construction.

$G$, $F$ and $H_j$ $(1\leqq j\leqq N)$ are assumed to satisfy the same conditions as in §2. Let $\tilde{Y}(x,t)=\hat{Y}(x,t)x^{D(t)}\exp((1/2)x^2G+xF^{(+)}(t))$ be the normalized formal matrix solution, and $Y_l(x,t)$ $(1\leqq l\leqq 5)$ the normalized matrix solution at $x=\infty$ for (1.2) in the sence of [6]. Here $F^{(+)}$ denotes the diagonal part of $F$. Near $x=a_j$, equation (1.2) has a fundamental solution matrix $Y_{a_j}(x,t)$ of the same form as (2.2). We define the Stokes multipliers $C_l$ $(1\leqq l\leqq 4)$ and the connection matrix $Q_j$ $(1\leqq j\leqq N)$ by

$$(3.1)\qquad Y_{l+1}=Y_lC_l,$$

$$(3.2)\qquad Y_1=Y_{a_j}Q_j.$$

We set the deformation properties as follows:

$$\text{(DP.I)}\qquad dD=0,\quad dC_l=0,\quad 1\leqq l\leqq 4,$$

$$\text{(DP.II)}\qquad d(Q_j^{-1}L_jQ_j)=0,\quad (I-J)dQ_j\cdot Q_j^{-1}J=0,\quad 1\leqq j\leqq N.$$

**Theorem 2.** *The deformation properties* (DP.I)–(DP.II) *hold if and only if* $G$, $F$ *and* $H_j$ $(1\leqq j\leqq N)$ *satisfy the following non-linear system*

$$(3.3)\qquad dP=[\Omega,P],\qquad d\Omega=\Omega\wedge\Omega,$$

*where* $\Omega=x^2\Phi+x\Psi+\Theta$ *is a matrix of* 1*-forms in* $t$ *given by*

$$\begin{aligned}(3.4)\quad & \Phi=(1/2)dG,\qquad \Psi=dF^{(+)}+\{\Phi,F\}_G\\ & \Theta=\{\Phi,\textstyle\sum_{j=1}^{N}H_j\}_G+\{\Psi,F\}_G\\ & \qquad +1/2\operatorname{diag}(f_{12}f_{21}d(1/(g_1-g_2)),f_{21}f_{12}d(1/(g_2-g_1))).\end{aligned}$$

*The above system* (3.3) *is equivalently rewritten into the following non-linear system,*

$$\begin{aligned}(3.5)\qquad & dF=\Psi+[\Theta,F]+\textstyle\sum_{j=1}^{N}a_j[\Phi,H_j]+\sum_{j=1}^{N}[\Psi,H_j]\\ & dH_j=[\Omega|_{x=a_j},H_j],\qquad 1\leqq j\leqq N.\end{aligned}$$

We remark that $G$ and $F^{(+)}$ can be regarded as independent variables.

**§4. The proof of Theorem 1.** First we prove the necessity of (2.5)–(2.6). Put $\Omega=dY_1\cdot Y_1^{-1}$. Let us assume that (DP.I)–(DP.III) hold. Then, by the same argument as in [6], we show that $\Omega$ has a local expansion at $x=\infty,0$

(4.1) $$\Omega = d\hat{Y}\cdot\hat{Y}^{-1} + x(\hat{Y}dG\hat{Y}^{-1}) \qquad \text{at } x=\infty,$$

(4.2) $$\Omega = dK\cdot K^{-1} + K\{d\hat{Z}\cdot\hat{Z}^{-1} - x^{-1}(\hat{Z}d\hat{E}\hat{Z}^{-1})K^{-1} \qquad \text{at } x=0.$$

By virtue of (2.3), we know further that

(4.3) $$\begin{aligned}\Omega = {} & dT_j\cdot T_j^{-1} + T_j(x-a_j)^J d\Phi_j\cdot\Phi_j^{-1}(x-a_j)^{-J}T_j^{-1} \\ & + T_j(x-a_j)^J\Phi_j\{dQ_j\cdot Q_j^{-1} + (dL_j + [L_j, dQ_j\cdot Q_j^{-1}])\log(x-a_j) \\ & - (dL_j\cdot L_j + L_j dQ_j\cdot Q_j^{-1}L_j)\log^2(x-a_j)\}\Phi_j^{-1}(x-a_j)^{-J}T_j^{-1} \\ & \text{at } x=a_j\ (1\leqq j\leqq N).\end{aligned}$$

It follows from (DP.III) that the logarithmic term and the residue of (4.3) vanish. Hence $\Omega$ is holomorphic at $x=a_j$ $(1\leqq j\leqq N)$ and has simple poles at $x=\infty, 0$. We obtain (2.5) as the integrability condition of the extended system $PY_1=0$, $dY_1=\Omega Y_1$, and also (2.6) by comparing the constant term of (4.1) with the one of (4.2). Next we show the converse. Suppose that the coefficient matrices of (1.1) satisfy the non-linear system (2.5)–(2.6). By a similar argument as [6], we show that $dD_\infty = dD_0 = 0$, and that $dY_l = \Omega Y_l$, $dZ_l = (K^{-1}\Omega K - K^{-1}dK)Z_l$ $(1\leqq l\leqq 3)$. Hence it follows that the Stokes multipliers $C_l^{(\infty)}$, $C_l^{(0)}$ $(1\leqq l\leqq 2)$ do not change. And (2.6) implies that $dQ_0=0$. Since $Y_1$ satisfies the extended system $PY_1=0$, $dY_1=\Omega Y_1$, the local monodromy at $x=a_j$ is preserved, i.e. $d\exp(2\pi i Q_j^{-1}L_jQ_j)=0$. Noting $(Q_j^{-1}L_jQ_j)^2=0$, we obtain $d(Q_j^{-1}L_jQ_j)=0$ $(1\leqq j\leqq N)$. We note also that the residue of $\Omega = dY_j\cdot Y_j^{-1}$ at $x=a_j$ vanishes, i.e. $(I-J)dQ_j\cdot Q_j^{-1}J=0$, for $\Omega$ is holomorphic there. This completes the proof.

**Remark.** It is easily seen that, by choosing an appropriate $T_j$, $Q_j$ itself is independent of the parameters under the deformation.

**§ 5. Application to $N$-soliton solutions of the sine-Gordon equation.** In [4], Date established a *direct construction method of multi-soliton solution*. It is well known that the sine-Gordon equation, $u_{\xi\eta} + \sin u = 0$, is the compatibility condition of the system of differential equations:

(5.1) $$\begin{aligned}&\left(\frac{\partial}{\partial\xi} - \frac{ix}{2}\begin{bmatrix} & 1\\ 1 & \end{bmatrix} - \frac{i}{2}\begin{bmatrix} u_\xi & \\ & -u_\xi\end{bmatrix}\right)Y=0,\\ &\left(\frac{\partial}{\partial\eta} - \frac{ix^{-1}}{2}\begin{bmatrix} & e^{iu}\\ e^{-iu} & \end{bmatrix}\right)Y=0.\end{aligned}$$

First we construct a matrix solution of (5.1) satisfying the following conditions:

(5.2) $$Y(x,\xi,\eta) = \hat{Y}(x,\xi,\eta)x^N \exp\left\{\frac{i}{2}\left(x\begin{bmatrix}\xi & \\ & -\xi\end{bmatrix} + x^{-1}\begin{bmatrix}\eta & \\ & -\eta\end{bmatrix}\right)\right\}$$

where $\hat{Y} = \begin{bmatrix} 1 & (-)^N \\ 1 & -(-)^N\end{bmatrix} + \sum_{j=1}^{N}\begin{bmatrix} y_{1,j}(\xi,\eta) & (-)^{N-j}y_{1,j}(\xi,\eta) \\ y_{2,j}(\xi,\eta) & -(-)^{N-j}y_{2,j}(\xi,\eta)\end{bmatrix}x^{-j}$,

(5.3) $$Y(\alpha_j,\xi,\eta)\begin{bmatrix} 1 \\ -c_j\end{bmatrix} = 0, \qquad Y(-\alpha_j,\xi,\eta)\begin{bmatrix} -c_j \\ 1\end{bmatrix} = 0.$$

Here $c_j$ and $\alpha_j$ are non-zero constants such that $\alpha_j \neq \alpha_k$ for $j\neq k$ and $\alpha_j \neq -\alpha_k$ for any $j, k$. Then (5.2) and (5.3) uniquely determine the entries of $Y(x,\xi,\eta)$.

**Proposition** (Date [4], Okamoto [7]). *Under the assumptions* (5.2)–(5.3), $Y(x,\xi,\eta)$ *satisfies the following equation*:

(5.4) $$dY=\Omega Y$$

*where*

$$\Omega=\left(\frac{ix}{2}\begin{bmatrix} & 1\\ 1 & \end{bmatrix}+\frac{i}{2}(y_{1,N-1}-y_{2,N-1})\begin{bmatrix}1 & \\ & -1\end{bmatrix}\right)d\xi+\frac{ix^{-1}}{2}\begin{bmatrix} & y_{1,0}/y_{2,0}\\ y_{2,0}/y_{1,0} & \end{bmatrix}d\eta.$$

*Here $d$ denotes the exterior differentiation with respect to $\xi$ and $\eta$. In other words, $Y(x,\xi,\eta)$ is a solution of* (5.1) *by the identification $e^{iu}=y_{1,0}/y_{2,0}$, $u_\xi=y_{1,N-1}-y_{2,N-1}$. And $u=i\log(y_{1,0}/y_{2,0})$ is an $N$-soliton solution of the sine-Gordon equation.*

Next we search for the $x$-equation satisfied by $Y(x,\xi,\eta)$. Setting $Y_\infty(x,\xi,\eta)=\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}Y(x,\xi,\eta)$, we know that $Y_\infty(x,\xi,\eta)$ solves the following equation:

(5.5) $$\frac{\partial Y}{\partial X}=\left\{x^{-2}E+x^{-1}F+G+\sum_{j=1}^{N}\left(\frac{H_{\alpha_j}}{x-\alpha_j}+\frac{H_{-\alpha_j}}{x+\alpha_j}\right)\right\}Y$$

where

$$G=\frac{i\xi}{2}\begin{bmatrix}1 & \\ & -1\end{bmatrix},\qquad E=K\tilde{E}K^{-1},\qquad E=-\frac{i\eta}{2}\begin{bmatrix}1 & \\ & -1\end{bmatrix}$$

$$K=\frac{1}{2}\begin{bmatrix}y_{1,0}+y_{2,0} & y_{1,0}-y_{2,0}\\ y_{1,0}-y_{2,0} & y_{1,0}+y_{2,0}\end{bmatrix},$$

$$F+\sum_{j=1}^{N}(H_{\alpha_j}+H_{-\alpha_j})=\begin{bmatrix}N & \frac{i\xi}{2}(y_{1,N-1}-y_{2,N-1})\\ \frac{i\xi}{2}(y_{1,N-1}-y_{2,N-1}) & N\end{bmatrix}.$$

We note that the eigenvalues of $H_{\pm\alpha_j}$ are 0 and 1, and that $x=\pm\alpha_j$ are apparent singular points. We investigate the global connection structure of (5.5). Firstly, we should observe that the Stokes multipliers of $Y_\infty$ around the infinity, $C_l^{(\infty)}$ $(1\leqq l\leqq 2)$ are all trivial, and that the formal monodromy $D^{(\infty)}=N$. Because the normalized solution of (5.5) around the origin is given by $Y_\infty=KZ_0$, the Stokes multipliers and the formal monodromy around the origin are all trivial. Next we introduce invertible matrices $T_{\pm\alpha_j}$, $Q_{\pm\alpha_j}$ $(1\leqq j\leqq N)$ as follows:

(5.6) $$H_{\pm\alpha_j}=T_{\pm\alpha_j}\operatorname{diag}(0,1)T_{\pm\alpha_j}^{-1},\qquad Y_\infty=T_{\pm\alpha_j}Y_{\pm\alpha_j}Q_{\pm\alpha_j}.$$

Here $Y_{\pm\alpha_j}$ is the normalized solution of (5.5) around $x=\pm\alpha_j$, expressed as $Y_{\pm\alpha_j}=(x\mp\alpha_j)\Phi_{\pm\alpha_j}(x\mp\alpha_j)^{L_{\pm\alpha_j}}$, where $L_{\pm\alpha_j}=\begin{bmatrix}0 & \\ l_{\pm\alpha_j} & 0\end{bmatrix}$, and $\Phi_{\pm\alpha_j}$ is holomorphic near $x=\pm\alpha_j$, and $\Phi_{\pm\alpha_j}(\pm\alpha_j)=I$. In the present argument, it is clear that $l_{\pm\alpha_j}=0$, because logarithmic terms are absent in $Y_\infty$. Moreover, by choosing an appropriate $T_{\pm\alpha_j}$, it is shown that $Q_{\pm\alpha_j}=\begin{bmatrix}1 & c_j^{\pm1}\\ & 1\end{bmatrix}$. As we have shown above, the deformation properties in the sense of

§2 hold. Then Theorem 1 asserts that $Y_\infty(x, \xi, \eta)$ should satisfy the equation $dY_\infty = \tilde{\Omega} Y_\infty$, where $\tilde{\Omega}$ is determined in accordance with the formula (2.7). We find it to be the same as $\Omega$ in (5.4).

Summing up, we obtain the following

**Theorem 3.** *The equation* (5.5) *is deformed with keeping the properties in* §2. *Therefore* $K$, $F$, *and* $H_{\pm\alpha_j}$ *satisfy the deformation equations* (2.8), *where* $H_j$ *are replaced by* $H_{\pm\alpha_j}$. *These equations characterize N-soliton solutions of the sine-Gordon equation.*

## References

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