1. A Note on Mikusiński's Operational Calculus

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§ 1. **Introduction.** In [ii], one of the present authors gave a simplified derivation of Mikusiński's operational calculus [i] without appealing to Titchmarsh's theorem concerning the vanishing of the convolution of two continuous functions defined on $[0, \infty)$.

The purpose of the present note is to give a further simplification of [ii] to the effect that we can derive the operational calculus directly from the ring C_H in [ii] without introducing the ring C_P in [ii]. For the sake of convenience for the reader, we shall begin with the definition of the ring C_H.

§ 2. **The ring C_H.** We denote by C the totality of complex-valued continuous functions defined on $[0, \infty)$. We denote such a function by $\{f(t)\}$ or simply by f, while $f(t)$ means the value at t of the function f. For $f, g \in C$ and $\alpha, \beta \in \boldsymbol{K}$ (=the complex number field) we define

(1) $$\alpha f+\beta g=\{\alpha f(t)+\beta g(t)\} \quad \text{and} \quad fg=\left\{\int_0^t f(t-s)g(s)ds\right\}.$$

Then C is a commutative ring with respect to the above addition and multiplication over the coefficient field $\boldsymbol{K}$.

We shall denote by h (l in [i]) the constant function $\{1\} \in C$ so that we have

(2) $$h^n=\left\{\frac{t^{n-1}}{(n-1)!}\right\} \qquad (n=1, 2, \cdots),$$

and

(3) $$hf=\left\{\int_0^t f(s)ds\right\} \qquad \text{for } f \in C,$$

i.e. h behaves as an operation of integration. Then we have the following fairly trivial

Proposition 1. *For $k \in \boldsymbol{H}=\{k\,;\, k=h^n\ (n=1, 2, \cdots)\}$ and $f \in C$, the equation $kf=0$ implies that $f=0$, where 0 denotes $\{0\} \in C$.*

Therefore we can construct the commutative ring C_H of fractions:

(4) $$C_H=\left\{\frac{f}{k}\,;\ f \in C \text{ and } k \in \boldsymbol{H}\right\}$$

where the equality is defined by

(5) $$\frac{f}{k}=\frac{f'}{k'} \qquad \text{if and only if } k'f=kf',$$

and the addition and multiplication are defined through

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$$(6)\qquad \frac{f}{k}+\frac{f'}{k'}=\frac{k'f+kf'}{kk'} \quad\text{and}\quad \frac{f}{k}\,\frac{f'}{k'}=\frac{ff'}{kk'},$$

respectively.

By identifying $f\in C$ with $kf/k\in C$, the ring C can be isomorphically embedded as a subring of the ring C_H. For any complex number α, we define

$$(7)\qquad [\alpha]=\frac{\{\alpha\}}{h}\in C_H.$$

Then we have, for $\alpha,\beta\in \boldsymbol{K}$, $f\in C$ and $k\in \boldsymbol{H}$,

$$(7)'\qquad \begin{aligned} &[\alpha]+[\beta]=[\alpha+\beta], && [\alpha][\beta]=[\alpha\beta],\\ &[\alpha]f=\alpha f=\{\alpha f(t)\}, && [\alpha]\frac{f}{k}=\frac{\{\alpha f(t)\}}{k}=\frac{\alpha f}{k}.\end{aligned}$$

Hence $[\alpha]$ can be identified with the complex number α, not with $\{\alpha\}$, and we see that the effect of the multiplication by $[\alpha]$ is exactly the $\boldsymbol{\alpha}$**-times multiple.** In particular [1] may be identified with the **multiplicative unit** I of C_H:

$$(8)\qquad I=\frac{h^n}{h^n}\qquad (n=1,2,\cdots).$$

We then define

$$(9)\qquad s=\frac{h^n}{h^{n+1}}\in C_H\qquad (n=0,1,2,\cdots;\ h^0=I)\text{ so that } sh=I.$$

Proposition 2. *If both f and its derivative f' belong to C, then we have*

$$(10)\qquad f'=sf-f(0),\qquad \textit{where } f(0)=[f(0)],$$

that is, s behaves as an **operation of differentiation.**

Proof. Clear from (9) and Newton's formula

$$hf'=\left\{\int_0^t f'(s)ds\right\}=\{f(t)-f(0)\}=f-[f(0)]h.$$

Corollary. *For n-times continuously differentiable function $f\in C$,*

$$(10)'\qquad \begin{aligned} f^{(n)}&=s^nf-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0),\\ &\textit{where } f^{(j)}(0)=[f^{(j)}(0)].\end{aligned}$$

Proposition 3. *For any $\alpha\in K$ and for any positive integer n, the element*

$$(s-\alpha)^n=(s-[\alpha])^n=\frac{(I-[\alpha]h)^n}{h^n}\in C_H$$

admits a uniquely determined **multiplicative inverse** *in C_H given by*

$$(11)\qquad \frac{I}{(s-\alpha)^n}=\left\{\frac{t^{n-1}}{(n-1)!}e^{\alpha t}\right\}=n\text{-times multiplication of }\{e^{\alpha t}\}.$$

Proof. We have $(s-\alpha)\{e^{\alpha t}\}=I$ by (10) and so (11) is easily obtained.

§ 3. The operational calculus. Consider the following Cauchy problem for linear ordinary differential equation with coefficients $\in \boldsymbol{K}$:

$$(12)\qquad \begin{cases}\alpha_n y^{(n)}+\alpha_{n-1}y^{(n-1)}+\cdots+\alpha_0 y=f\in\mathcal{C} & (\alpha_n\neq 0),\\ y(0)=\gamma_0,\ y'(0)=\gamma_1,\ \cdots,\ y^{(n-1)}(0)=\gamma_{n-1}.\end{cases}$$

By (10)′, we can rewrite (12) into equation in $\mathcal{C}_H$:

$$(12)'\qquad \begin{aligned}&(\alpha_n s^n+\alpha_{n-1}s^{n-1}+\cdots+\alpha_0)y=f+\beta_{n-1}s^{n-1}+\beta_{n-2}s^{n-2}+\cdots+\beta_0,\\ &\beta_m=\alpha_{m+1}\gamma_0+\alpha_{m+2}\gamma_1+\cdots+\alpha_n\gamma_{n-m-1}\qquad (m=0,1,2,\cdots,n-1).\end{aligned}$$

Since the polynomial ring of polynomials in s with coefficients in K is free from zero factors, we can define rational functions

$$F_1=\frac{I}{\alpha_n s^n+\cdots+\alpha_0}\quad\text{and}\quad F_2=\frac{\beta_{n-1}s^{n-1}+\cdots+\beta_0}{\alpha_n s^n+\cdots+\alpha_0}$$

and obtain their partial fraction expressions:

$$(13)\qquad F_1=\sum_j\sum_{k=1}^{m_j}\frac{c_{jk}I}{(s-r_j)^k}\quad\text{and}\quad F_2=\sum_j\sum_{k=1}^{m_j}\frac{d_{jk}I}{(s-r_j)^k},$$

where c_{jk} and d_{jk} belong to K and r_j are roots of the algebraic equation $\alpha_n z^n+\cdots+\alpha_0=0$ so that $\sum_j m_j=n$. As was proved in (11), F_1 and F_2 given in (13) belong to $\mathcal{C}\subset\mathcal{C}_H$ so that we obtain, from (12)′, the solution of (12):

$$\{y(t)\}=\sum_j\sum_{k=1}^{m_j}c_{jk}\left\{\frac{t^{k-1}}{(k-1)!}e^{r_j t}\right\}\{f(t)\}+\sum_j\sum_{k=1}^{m_j}d_{jk}\left\{\frac{t^{k-1}}{(k-1)!}e^{r_j t}\right\}$$

In this way, Mikusiński's operational calculus can be derived without appealing to Titchmarsh's theorem nor to the ring $\mathcal{C}_P$ in [ii], that is, the totality of fractions f/p of the form $f(\in\mathcal{C})$ over non-zero polynomial p (in t) with cofficients $\in K$.

References

[i] Jan Mikusiński: Operational Caluclus. Pergamon Press (1959).

[ii] Shûichi Okamoto: A simplified derivation of Mikusinski's operational calculus. Proc. Japan Acad., **55**A(1), 1–5 (1979).