23. *Rational Maps to Varieties of Hyperbolic Type**)

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§ 0. **Introduction.** In this paper we prove a finiteness theorem for the set of dominant rational maps to a variety of hyperbolic type. Let k be an algebraically closed field of characteristic zero. In this paper we assume all varieties are defined over k.

First we recall the definition of the Kodaira dimension. Let X be a smooth algebraic variety, then by Nagata and Hironaka there exists a complete smooth algebraic variety $\bar{X}$ such that $D_X := \bar{X} - X$ is a divisor with normal crossings. Let K_X be the canonical divisor of $\bar{X}$ and Φ_m the rational map of $\bar{X}$ which is associated with the linear system $|m(K_X+D_X)|$.

Definition 1 ([3] and [4]). The logarithmic Kodaira dimension $\bar{\kappa}(X)$ of X is

$$\begin{cases} \sup_{m>0} \dim \Phi_m(\bar{X}), & \text{if } |m(K_X+D_X)| \neq (0) \text{ for some } m \in N, \\ -\infty & \text{, if } |m(K_X+D_X)| = (0) \text{ for every } m \in N. \end{cases}$$

If X is complete, $\bar{\kappa}(X)$ is denoted by $\kappa(X)$ and is called the Kodaira dimension of X. X is said to be of elliptic type, of parabolic type, and of hyperbolic type, if $\bar{\kappa}(X) = -\infty$, 0, and $\dim(X)$, respectively. Algebraic varieties of hyperbolic type are also called of general type.

This notion of hyperbolicity is different from that of [5]. But it is known that a smooth algebraic variety of hyperbolic type is measure-hyperbolic in the sense of [5] (cf. [8]). The Kodaira dimension is an important bi-rational invariant to classify algebraic varieties (cf. [10]).

Definition 2. Let X and Y be algebraic varieties. A rational map $f: X \rightarrow Y$ is said to be a strictly rational map, if there exists a proper bi-rational morphism $\pi: X' \rightarrow X$ such that $f \circ \pi$ is a morphism. f is said to be dominant, if $\dim (f \circ \pi)(X') = \dim (Y)$.

Our main theorem is as follows:

Theorem. *Let X be a smooth algebraic variety and Y a smooth algebraic variety of hyperbolic type. Then the set of dominant strictly rational maps of X to Y is finite.*

The following varieties are examples of varieties of hyperbolic type.

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*) This is a shorter version of the master thesis submitted by the author in February 1977 to the University of Tokyo.

Example 1. Let D be a hypersurface with normal crossings of $\boldsymbol{P}^n$ with degree $\geqq n+2$. Then $\boldsymbol{P}^n - D$ is of hyperbolic type.

Example 2. Let $\mathscr{D}$ be a bounded symmetric domain and Γ a discrete arithmetic subgroup of the group of bi-holomorphic automorphism of $\mathscr{D}$. Then $\mathscr{D}/\Gamma$ is an algebraic variety (cf. [1]). If, moreover, Γ has no torsion element, then $\mathscr{D}/\Gamma$ is of hyperbolic type (cf. [9]).

Kobayashi and Ochiai proved the following theorem in [6].

Theorem. *Let X be a compact complex manifold and Y a compact complex manifold of general type. Then the set of dominant meromorphic maps of X to Y is finite.*

Moreover Iitaka and Sakai proved the following theorem in [4] and [8].

Theorem. *Let X be a smooth algebraic variety of hyperbolic type. Then the set of strictly bi-rational maps: $X \to X$ is finite.*

Our theorem can be seen as a generalization of these theorems.

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§ 1. Preliminary. Let X_i $(i=1,2)$ be a smooth algebraic variety which is an open subset of a complete smooth algebraic variety $\bar{X}_i$ such that $D_i := \bar{X}_i - X_i$ is a divisor with normal crossings. The sheaf of germs of logarithmic q-forms on $\bar{X}_i$ along D_i is defined as in [2], which we denote by $\Omega^q(\log D_i)$.

Lemma 1 (cf. [4]). *Let $f: X_1 \to X_2$ be a strictly rational map and m a positive integer. And let $\bar{f}: \bar{X}_1 \to \bar{X}_2$ be the extension of f. Then if $\omega \in \Gamma(\bar{X}_2, (\Omega^q(\log D_2))^{\otimes m})$, $\bar{f}^*(\omega) \in \Gamma(\bar{X}_1, (\Omega^q(\log D_1))^{\otimes m})$.*

Let X and Y be as in the Theorem. In §§ 1 and 2, we assume that $\dim(X) = \dim(Y) = n$. Let $\bar{X}$ (resp. $\bar{Y}$) be a complete algebraic variety which contains X (resp. Y) as its open subset such that $D_X := \bar{X} - X$ (resp. $D_Y := \bar{Y} - Y$) is a divisor with normal crossings.

To prove the theorem we may assume that $\bar{X}$ and $\bar{Y}$ are projective. If it is not the case, let $\pi_X: \bar{X}' \to \bar{X}$ (resp. $\pi_Y: \bar{Y}' \to \bar{Y}$) be a bi-rational morphism such that $\bar{X}'$ (resp. $\bar{Y}'$) is projective and $\pi_X^{-1}(D_X)$ (resp. $\pi_Y^{-1}(D_Y)$) is a divisor with normal crossings. Then $\pi_Y^{-1}(Y)$ is of hyperbolic type by Lemma 1, and if $f: X \to Y$ is a strictly rational map, $(\pi_{Y|\pi_Y^{-1}(Y)})^{-1} \circ f \circ \pi_{X|\pi_X^{-1}(X)}$ is also a strictly rational map, since a composition of strictly rational maps is a strictly rational map. We replace X (resp. Y) by $\pi_X^{-1}(X)$ (resp. $\pi_Y^{-1}(Y)$) and $\bar{X}$ (resp. $\bar{Y}$) by $\bar{X}'$ (resp. $\bar{Y}'$).

Let $K_{\bar{X}}$ (resp. $K_{\bar{Y}}$) be the canonical divisor of $\bar{X}$ (resp. $\bar{Y}$). The linear system $|m(K_{\bar{X}} + D_X)|$ (resp. $|m(K_{\bar{Y}} + D_Y)|$) is canonically identified with $\Gamma(\bar{X}, (\Omega^n(\log D_X))^{\otimes m})$ (resp. $\Gamma(\bar{Y}, (\Omega^n(\log D_Y))^{\otimes m})$. Let $f: X \to Y$ be a strictly rational map and $\bar{f}: \bar{X} \to \bar{Y}$ its extension. Then the following condition (*) holds by Lemma 1.

(*) If $\omega \in |m(K_{\bar{Y}} + D_Y)|$, $\bar{f}^*(\omega) \in |m(K_{\bar{X}} + D_X)|$.

Let R be the set of dominant rational maps of $\bar{X}$ to $\bar{Y}$ which satisfy the condition (*). It suffices to prove that R is finite. If $f \in R$, $f^*: |m(K_{\bar{Y}}+D_Y)| \to |m(K_{\bar{X}}+D_X)|$ is injective, since f is dominant. Hence if $R \neq \emptyset$, X is also of hyperbolic type. We assume that $R \neq \emptyset$.

There exists a positive integer m such that the rational map of $\bar{X}$ which is associated with $|m(K_{\bar{X}}+D_X)|$ is bi-rational to its image. Furthermore we can take a positive integer m so that the following condition (**) also holds (cf. [6]).

(**) There exists an effective divisor C on $\bar{Y}$ such that the linear system $|m(K_{\bar{Y}}+D_Y)-C|$ is very ample.

We fix this m throughout §§ 1 and 2. Put

$$V_X := |m(K_{\bar{X}}+D_X)|,$$
$$V_Y := |m(K_{\bar{Y}}+D_Y)-C|,$$
$$\boldsymbol{H} := \mathrm{Hom}\,(V_Y, V_X)^{\vee} \quad (^{\vee} \text{ means the dual}).$$

Let i_X be the rational map of $\bar{X}$ to $\boldsymbol{P}(V_X)$ which is associated with V_X and i_Y the embedding of $\bar{Y}$ to $\boldsymbol{P}(V_Y)$ which is associated with V_Y. If $f \in R$, $f^*: V_Y \to V_X$ determines a point $f^{\vee}$ of $\boldsymbol{P}(\boldsymbol{H})$, since f^* is not zero. There follows the commutative diagram below.

$$\begin{array}{ccc} \boldsymbol{P}(V_X) & \xrightarrow[f^{\vee}]{} & \boldsymbol{P}(V_Y) \\ \uparrow{\scriptstyle i_X} & & \uparrow{\scriptstyle i_Y} \\ \bar{X} & \xrightarrow[f]{} & \bar{Y} \end{array} .$$

Lemma 2. *Let $f, g \in R$. If $f^{\vee}=g^{\vee}$, then $f=g$.*

Proof. Since i_X is bi-rational and i_Y is an embedding.

§ 2. Proof of the finiteness of $\boldsymbol{R}$. Let $F_0: \boldsymbol{P}(V_X)\times \boldsymbol{P}(\boldsymbol{H}) \to \boldsymbol{P}(V_Y)$ be the rational map which is determined by the morphism: $V_X \times \boldsymbol{H} \to V_Y$. For a rational map f we denote by Ind (f) the set of points of indeterminacy of f. Let T be an algebraic variety and $G: \bar{X}\times T \to \bar{Y}$ a rational map which satisfies the following condition (***).

(***) $\bar{X}\times\{t\} - \mathrm{Ind}\,(G)$ is a non-empty open subset of $\bar{X}$ for every $t \in T$.

Then we denote by $\bar{X}_t$ $\bar{X}\times\{t\}$ and by G_t the restriction of G to $\bar{X}_t$. If $h \in \boldsymbol{P}(\boldsymbol{H})$, Ind $(F_0)\cap \boldsymbol{P}(V_X)\times\{h\}$ is a linear subspace of $\boldsymbol{P}(V_X)\times\{h\}$. Since $i_X(\bar{X})$ is not contained in a hyperplane of $\boldsymbol{P}(V_X)$, the restriction of F_0 to $i_X(\bar{X})\times \boldsymbol{P}(\boldsymbol{H})$ satisfies the similar condition to (***). $F := F_0\circ(i_X \times id_{\boldsymbol{P}(\boldsymbol{H})})$ is a rational map of $\bar{X}\times \boldsymbol{P}(\boldsymbol{H})$ to $\boldsymbol{P}(V_Y)$ satisfying (***).

The following two lemmas can be proved easily.

Lemma 3. *Let H_1 be the subset of $\boldsymbol{P}(\boldsymbol{H})$ such that $h \in H_1$ if and only if $F_h(\bar{X}_h) \subset i_Y(\bar{Y})$. Then H_1 is a closed subset of $\boldsymbol{P}(\boldsymbol{H})$.*

Lemma 4. *Let H_2 be the subset of H_1 such that $h \in H_1$ belongs to H_2 if and only if F_h is dominant. Then H_2 is an open subset of H_1.*

Lemma 5. *Let T be an algebraic variety and $G: \bar{X}\times T \to \bar{Y}$ a strictly rational map satisfying (***) such that G_t is dominant for* *every $t \in T$. And let ω be an element of V_Y and T_ω the subset of T such that $t \in T$ belongs to T_ω if and only if $G_t^*(\omega) \in V_X$. Then T_ω is a closed subset of T.*

Proof. Let $p : \tilde{T} \to T$ be a desingularization of T and $\tilde{G} = G \circ (id_{\bar{X}} \times p)$. It sufficies to prove that $\tilde{T}_\omega$ is closed subset of T, since p is proper. Let $\bigwedge^n(T^*_{\bar{X}\times\tilde{T}})$ and $\bigwedge^n(T^*_{\tilde{T}})$ be the vector bundle of n-forms on $\bar{X} \times \tilde{T}$ and $\tilde{T}$ respectively. There exists an exact sequence of vector bundles;

$$0 \longrightarrow p_2^*(\bigwedge^n(T^*_{\tilde{T}})) \longrightarrow \bigwedge^n(T^*_{\bar{X}\times\tilde{T}}) \xrightarrow{\pi} \bigwedge^n(T^*_{\bar{X}\times\tilde{T}/\tilde{T}}) \longrightarrow 0.$$

$\bigwedge^n(T^*_{\bar{X}\times\tilde{T}/\tilde{T}})$ is isomorphic to the pull back of the canonical line bundle of $\bar{X}$ by $p_1 : \bar{X} \times \tilde{T} \to \bar{X}$.

$\tilde{G}^*(\omega)$ is a rational section of $(\bigwedge^n(T^*_{\bar{X}\times\tilde{T}}))^{\otimes m}$ and $\pi^{\otimes m}(\tilde{G}^*(\omega))$ is a rational section of $(\bigwedge^n(T^*_{\bar{X}\times\tilde{T}/\tilde{T}}))^{\otimes m}$, where π is as in the sequence above. We may assume that $\omega \neq 0$. Let E and F be the divisor of zeros and poles of $\pi^{\otimes m}(\tilde{G}^*(\omega))$ respectively, and E_t and F_t the restriction of E and F to $\bar{X}_t$ respectively. Let $i_t : \bar{X}_t \to \bar{X} \times \tilde{T}$ be the inclusion. Since $\tilde{G}_t$ is dominant, $\bar{X}_t - \mathrm{Ind}(\tilde{G}) - \tilde{G}_t^{-1}(D_Y)$ is a non-empty open subset of $\bar{X}_t$. On $\bar{X}_t - \mathrm{Ind}(\tilde{G}) - \tilde{G}_t^{-1}(D_Y)$ it holds that $i_t^*(\pi^{\otimes m}(\tilde{G}^*(\omega))) = \tilde{G}_t^*(\omega)$. Hence $E_t, F_t \subsetneqq \bar{X}_t$.

Now we prove the assertion by an induction on $\dim(T)$. Let S be the subset of $\tilde{T}$ such that $t \in \tilde{T}$ belongs to S if and only if $\dim(E_t \cap F_t) \geqq n-1$. Then S is closed by the semi-continuity of $\dim(E_t \cap F_t)$. By the assumption of the induction, S_ω is a closed subset of S, hence of $\tilde{T}$.

By Lemma 6 below, F_t ($t \in \tilde{T}$) constitute a flat family of divisors. Hence there exists a morphism g of $\tilde{T}$ to the Hilbert scheme of $\bar{X}$ such that $g(t)$ is the point corresponding to F_t. Let $D_X = \sum_{i=1}^N D^i$ be the irreducible decomposition of D_X. And let $A = \{a = (a_1, a_2, \cdots, a_N)\,;\, a_i \in \boldsymbol{Z}$ and $0 \leqq a_i \leqq m\}$, and p_a ($a \in A$) the point of the Hilbert scheme of $\bar{X}$ which corresponds to the divisor $\sum_{i=1}^N a_i D^i$. Then it is easily seen that $\tilde{T}_\omega = g^{-1}(\{p_a\}_{a\in A}) \cup S_\omega$. Hence $\tilde{T}_\omega$ is closed in $\tilde{T}$.

Lemma 6 ([7, §20 F Corollary 1]). *Let A be a Noetherian ring, B a Noetherian A-algebra, M a finite B-module, and $f \in B$. Assume that*

(i) *M is A-flat.*

(ii) *For any maximal ideal P of B, f is $M/(P \cap A)M$-regular.*

Then f is M-regular and M/fM is A-flat.

Let $H_3 = \bigcap_{\omega \in V_Y} (H_2)_\omega$, then H_3 is a closed subset of H_2. R is mapped injectively to H_3 by the correspondence: $f \mapsto f^\vee$. Let $\bar{H}_3$ be the closure of H_3 in $\boldsymbol{P}(\boldsymbol{H})$, and $q : \tilde{H}_3 \to \bar{H}_3$ a desingularization of $\bar{H}_3$ such that $q^{-1}(\bar{H}_3 - H_3)$ is a divisor with normal crossings. And let $\tilde{F} = F|_{\bar{X}\times\bar{H}_3} \circ (id_{\bar{X}} \times q)$ be the rational map of $\bar{X} \times \tilde{H}_3$ to $\bar{Y}$.

Let H_4 be the subset of $\tilde{H}_3$ such that $h \in \tilde{H}_3$ belongs to H_4 if and only if $\tilde{F}_h(\bar{X}_h) \subset D_Y$. It is easily seen that H_4 is a closed subset of $\tilde{H}_3$. Let ω be an element of V_Y. If $h \notin H_4$, $\tilde{F}_h^*(\omega)$ is defined on $\bar{X}_h$ and coincides with $i_h^*(\pi^{\otimes m}(\tilde{F}^*(\omega)))$ on a non-empty open subset of $\bar{X}_h$ as in the proof of Lemma 5. $\tilde{F}_h^*(\omega)$ is not defined if $h \in H_4$. But we can prove that $i_h^*(\pi^{\otimes m}(\tilde{F}^*(\omega)))$ is defined and contained in V_X even if $h \in H_4$.

Lemma 7. $i_h^*(\pi^{\otimes m}(\tilde{F}^*(\omega))) \in V_X$.

Proof. By Lemma 1 and the definition of H_3, $\tilde{F}^*(\omega)$ is a logarithmic m-ple n-form on $\bar{X} \times \tilde{H}_3$ along $D_X \times \tilde{H}_3 \cup p_2^{-1}(H_4)$. Hence $\pi^{\otimes m}(\tilde{F}^*(\omega))$ is a logarithmic form along $D_X \times \tilde{H}_3$, hence $i_h^*(\pi^{\otimes m}(\tilde{F}^*(\omega)))$ is a logarithmic form along D_X.

Thus we constructed a linear map $f_h : V_Y \to V_X$ for $h \in \tilde{H}_3$, hence a morphism $\varphi : \tilde{H}_3 \to \boldsymbol{H}$ such that $\varphi(h) = f_h^\vee$. Since $\tilde{H}_3$ is complete and $\boldsymbol{H}$ is affine, Im (φ) is finite. If $h \notin H_4$, $\varphi(h)^\vee : V_Y \to V_X$ coinsides with $F_{q(h)}^* : V_Y \to V_X$. Therefore the image of the map ψ of H_3 to $\boldsymbol{H}$: $h \mapsto F_h^*$ is finite. Now it follows that H_3 is finite, since ψ is injective as in Lemma 2, which proves that R is finite.

§3. The case dim $(X) >$ dim (Y). In this section we prove the theorem in the case dim $(X) >$ dim (Y). Assume that there exist infinite dominant strictly rational maps f_i $(i = 1, 2, \cdots)$ of X to Y. Then by [6] there exists a subvariety X_1 of X such that all f_i $(i = 1, 2, \cdots)$ are defined and distinct on X_1, $f_{i|X_1}$ are dominant, and dim $(X_1) =$ dim (Y). This contradicts to the result of §2.

§4. In analytic categories. Let X be a complex manifold such that there exists a compact complex space which contains X as its Zariski open subset. Then the logarithmic Kodaira dimension of X is defined as in the case of algebraic varieties. Then by the similar method as before we can prove the following:

Theorem. *Let X be a complex manifold which X is a Zariski open subset of a Moishezon manifold $\bar{X}$ and Y a complex manifold which Y is a Zariski open subset of a compact complex manifold $\bar{Y}$. Then if Y is of hyperbolic type, the set of dominant meromorphic maps of X to Y which are extended to meromorphic maps of $\bar{X}$ to $\bar{Y}$ is finite.*

We need the assumption that $\bar{X}$ is a Moishezon manifold for the existence of an analytic subspace of X such as X_1 in §3.

A dominant holomorphic map of X to Y is extended to a meromorphic map of $\bar{X}$ to $\bar{Y}$ (cf. [8]). Hence the set of dominant holomorphic maps of X to Y is finite.

References

[1] Baily, W. L., Jr., and Borel, A.: Compactification of arithmetic quotients of bounded symmetric domains. Ann. of Math., 84, 442–528 (1966).

[2] Deligne, P.: Theorie de Hodge. II. Publ. Math. IHES, **40**, 5–58 (1973).
[3] Iitaka, S.: On D-dimension of algebraic varieties. J. Math. Soc. Japan, **23**, 356–373 (1971).
[4] ——: On logarithmic Kodaira dimension of algebraic varieties. Complex Analysis and Algebraic Geometry, Iwanami, Tokyo, 175–189 (1977).
[5] Kobayashi, S.: Hyperbolic Manifolds and Holomorphic Mappings. Marsel Dekker, New York (1970).
[6] Kobayashi, S., and Ochiai, T.: Meromorphic mappings onto complex spaces of general type. Invent. Math., **31**, 7–16 (1975).
[7] Matsumura, H.: Commutative Algebra. Benjamin, New York-Amsterdam (1970).
[8] Sakai, F.: Kodaira dimensions of complements of divisors. Complex Analysis and Algebraic Geometry, Iwanami, Tokyo, 239–257 (1977).
[9] Tsushima, R.: Hyperbolicity of quotients of bounded symmetric domains. Proc. Int. Symp. Algebraic Geometry, Kyoto, 1977, Kinokuniya, Tokyo, 687–691 (1978).
[10] Ueno, K.: Classification theory of algebraic varieties and compact complex spaces. Lecture Notes in Math., vol. 439, Springer, Berlin-Heidelberg-New York (1975).