22. A Note on the Large Sieve. III

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1. The purpose of the present note is to prove a large sieve version of a recent sieve result of Selberg [4] by combining his argument with that of our preceding note [1] of this series.

Before stating our results we have to introduce some conventions: For a prime p let $\Omega(p^\alpha)$ be a set of residues (mod p^α), and let us assume that $\Omega(p^\alpha)$ and $\Omega(p^\beta)$ are disjoint (mod p^β) whenever $0<\beta<\alpha$. For a composite d $\Omega(d)$ denotes the set of residues (mod d) arising from those of $\Omega(p^\alpha)$ with $p^\alpha \| d$ (the maximum power of p dividing d), and we write $n \in \Omega(d)$ to indicate that n (mod p^α) $\in \Omega(p^\alpha)$ for each $p^\alpha \| d$; so $n \in \Omega(1)$ for any n.

Following Selberg we put

$$\theta(p^\alpha)=1-\sum_{j=1}^{\alpha}|\Omega(p^j)|p^{-j},$$
$$g(d)=d^{-1}\prod_{p^\alpha\|d}\{|\Omega(p^\alpha)|\theta(p^\alpha)/\theta(p^{\alpha-1})\},$$

$|\Omega(p^\alpha)|$ being the cardinality of the set; here and in what follows we may assume $\theta(p^\alpha)\neq 0$ always. Also, if $d|r$, we put

$$t(r,d)=\prod_{\substack{p^\alpha\|r\\ p^\beta\|d}} t(p^\alpha,p^\beta), \qquad t^*(r,d)=\prod_{\substack{p^\alpha\|r\\ p^\beta\|d}} t^*(p^\alpha,p^\beta),$$

where $t(p^\alpha,p^\beta)=1$ if $\alpha=\beta$, $=|\Omega(p^\alpha)|p^{-\alpha}$ if $\beta=0$, and $=-|\Omega(p^\alpha)|(\theta(p^\beta)p^\alpha)^{-1}$ if $0<\beta<\alpha$; $t^*(p^\alpha,p^\beta)=1$ if $\alpha=\beta$, $=-|\Omega(p^\alpha)|(\theta(p^{\alpha-1})p^\alpha)^{-1}$ if $\beta=0$, and $=|\Omega(p^\alpha)|(\theta(p^{\alpha-1})p^\alpha)^{-1}$ if $0<\beta<\alpha$. Further $\Gamma_r(n,\Omega)$ stands for the sum

$$\sum_{\substack{u|r\\ n\in\Omega(u)}} t^*(r,u)$$

which is equal to $t^*(r,1)$ if $n \notin \Omega(p^\beta)$ for each $p^\beta|r$, ($\beta>0$).

Then our results are as follows:

Theorem. *Uniformly for any complex numbers a_n and for any $M,N,Q>0$, we have*

$$\sum_{\substack{qr\leq Q\\ (q,r)=1}}{}' \sum_{\chi(\mathrm{mod}\, q)}{}^* \frac{q}{\varphi(q)g(r)}\left|\sum_{M<n\leq M+N}\chi(n)\Gamma_r(n,\Omega)a_n\right|^2 \leq (N+Q^2)\sum_{M<n\leq M+N}|a_n|^2,$$

where φ is the Euler function, $\sum^*$ denotes a sum over primitive Dirichlet characters χ, and $\sum'$ indicates that r is restricted by $g(r)\neq 0$.

Corollary. *If $a_n=0$ whenever there exists a p^α such that $n\in\Omega(p^\alpha)$,*

$\alpha \geqq 1$, then we have

$$\sum_{\substack{qr\leqq Q\\(q,r)=1}} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \frac{q}{\varphi(q)} \prod_{p^\alpha \| r} \left(\frac{1}{\theta(p^\alpha)}-\frac{1}{\theta(p^{\alpha-1})}\right)\left|\sum_{M<n\leqq M+N}\chi(n)a_n\right|^2$$
$$\leqq (N+Q^2)\sum_{M<n\leqq M+N}|a_n|^2.$$

The function $\Gamma_r(n, \Omega)$ is obtained from the optimization procedure [4] of Selberg's weights λ_d for the sieve problem with the exclusion residues $\{\Omega(p^\alpha)\}$. And our theorem states that $\{\chi(n)\Gamma_r(n, \Omega)\}$ is a set of orthogonal pseudocharacters,[1] provided the conditions given there, the fact which can be easily generalized for any optimal Selberg weights (see also [2, Section 2]).

2. To prove the theorem we consider the dual form

$$D=\sum_{M<n\leqq M+N}\left|\sum_{\substack{qr\leqq Q\\(q,r)=1}} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \left(\frac{q}{\varphi(q)}\right)^{1/2}\chi(n)\Gamma_r(n, \Omega)b(r,\chi)\right|^2,$$

where $b(r, \chi)$ are arbitrary complex numbers. And we need following lemmas:

Lemma 1. *If $v|u$, then*

$$\sum_{\substack{\delta|u\\ \delta\equiv 0(\mathrm{mod}\, v)}} t^*(u,\delta)t(\delta,v)$$

is equal to 1 when $u=v$, and to 0 otherwise.

Lemma 2. *Let us put*

$$f(u,v)=\prod_{\substack{p^\alpha\|u\\p^\beta\|v}} f(p^\alpha,p^\beta),$$

where $f(p^\alpha, p^\beta)=|\Omega(p^\alpha)|p^{-\alpha}$ if $\alpha\beta(\alpha-\beta)=0$, and $=0$ otherwise. Then we have

$$f(u,v)=\sum_{\substack{\delta|u\\ \delta|v}} g(\delta)t(u,\delta)t(v,\delta).$$

Lemma 3. *For any complex numbers $c(u, h, \chi)$ and for any $M, N, Q>0$, we have*

$$\sum_{M<n\leqq M+N}\left|\sideset{}{^{**}}\sum \left(\frac{q}{\varphi(q)}\right)^{1/2}\chi(n)\exp\left(2\pi i\frac{h}{u}n\right)c(u,h,\chi)\right|^2$$
$$\leqq (N+Q^2)\sideset{}{^{**}}\sum |c(u,h,\chi)|^2,$$

where $\sum^{**}$ denotes the sum over $uq\leqq Q$, $(u, q)=1$; $1\leqq h\leqq u$, $(u, h)=1$; primitive χ (mod q).

Lemmas 1 and 2 are due to Selberg [4], and are immediate consequences from the definitions of functions relevant to those formulas. Lemma 3 can be reduced to the conventional additive large sieve inequality by considering the dual form.

3. Now we estimate D. From the definition of $\Gamma_r(n, \Omega)$ we have

$$D=\sum_{M<n\leqq M+N}\left|\sum_{\substack{n\in\Omega(u)\\ uq\leqq Q,(u,q)=1}} \sideset{}{^*}\sum_{\chi(\mathrm{mod}\, q)} \left(\frac{q}{\varphi(q)}\right)^{1/2}\chi(n)s(u,\chi)\right|^2,$$

where

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1) For this terminology see [3].

$$s(u,\chi)=\sum_{\substack{r\leqq Q/q\\ r\equiv 0(\mathrm{mod}\ u)\\ (r,q)=1}} b(r,\chi)t^*(r,u).$$

Then, as in [1], we express the characteristic function of the set of $n\in\Omega(u)$ as a trigonometrical sum, and we get

$$D=\sum_{M<n\leqq M+N}\left|{\sum}^{**}\left(\frac{q}{\varphi(q)}\right)^{1/2}\chi(n)\exp\left(2\pi i\frac{h}{u}n\right)y(u,h,\chi)\right|^2,$$

where ${\sum}^{**}$ is defined in Lemma 3, and

$$y(u,h,\chi)=\sum_{\substack{w\leqq Q/q\\ w\equiv 0(\mathrm{mod}\ u)\\ (w,q)=1}} s(w,\chi)w^{-1}\sum_{\substack{l=1\\ l\in\Omega(w)}}^{w}\exp\left(-2\pi i\frac{h}{u}l\right).$$

Hence by Lemma 3

$$D\leqq(N+Q^2)\,{\sum}^{**}|y(u,h,\chi)|^2.$$

Further, expanding out the squares and changing the order of summations in a suitable manner, we infer without difficulty that

$$D\leqq(N+Q^2)\sum_{\substack{d_1q\leqq Q\\ d_2q\leqq Q\\ (d_1d_2,q)=1}}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} s(d_1,\chi)\overline{s(d_2,\chi)}f(d_1,d_2),$$

where $f(d_1,d_2)$ is defined in Lemma 2. Thus by the same lemma

$$D\leqq(N+Q^2)\sum_{\substack{\delta q\leqq Q\\ (\delta,q)=1}}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} g(\delta)\Big|\sum_{\substack{d\equiv 0(\mathrm{mod}\ \delta)\\ (d,q)=1\\ d\leqq Q/q}} s(d,\chi)t(d,\delta)\Big|^2.$$

But the last sum over d is $b(\delta,\chi)$, because of Lemma 1. Therefore we have obtained

$$D\leqq(N+Q^2)\sum_{\substack{\delta q\leqq Q\\ (\delta,q)=1}}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\ q)} g(\delta)\,|b(\delta,\chi)|^2,$$

which is obviously equivalent to the assertion of the theorem.

References

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